Proceeding of

Workshop on Geometry and Topology around Young-Nam

edited by
Byung Hee An

November 2020

Preface

The first Workshop on Geometry and Topology aroung Young-Nam was held at *Kyungpook National University* in Daegu, Korea from November 19 to 21. There were 16 participants from Young-Nam area and one special guest from Incheon. Most of the participants are in early career such as students, post-docs, and assistant professors.

The academic program consisted of 10 indivisual lectures and this proceeding contains all articles which are either full paper versions or extended abstracts.

The workshop was mainly supported by National Research Foundation of Korea, several researcher programs led by Byung Hee An and Youngjin Bae, by Seung Yeop Yang, and by Juncheol Pyo. The workshop was also partly supported by the Conference Supporting Program of Kyungpook National University.

Last but not least, we thank all participants and speakers for making the workshop a great success. We hope this workshop will continue in future.

November 25, 2020
Byung Hee An, Kiryoung Chuyng, Juncheol Pyo and Seung Yeop Yang.

Workshop on Geometry and Topology around Young-Nam

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List of Organizers

Workshop on Geometry and Topology around Young-Nam

Kyungpook National University, Daegu, Korea
November 19 – 21, 2020

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- Byung Hee An (Kyungpook National University)
- Kiryong Chung (Kyungpook National University)
- Juncheol Pyo (Pusan National University)
- Yang Seung Yeop (Kyungpook National University)

Sponsered by

- Department Of Mathematics Education, Kyungpook National University
- National Research Foundation of Korea
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Lectures

CONFIGURATION SPACES OF GRAPHS

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Abstract. The configuration space of a topological space is a collection of a number of points without collision. It can be used to define the braid group over any topological space. In this article, we focus on the configuration spaces of graphs and discuss how we can extract geometric or combinatorial invariants of the underlying graphs from homotopy invariants of their configuration spaces.

1. Preliminaries

Let X be a topological space which is locally compact Hausdorff. Basically, the space X is a collection of points in X, which sounds like a tautology.

$$X = \{x \mid x \in X\}$$

However, one may regard each point $x \in X$ as the map f_x from the singleton set $\{1\}$ to X such that the image of the element 1 under f_x is precisely x.

$$X \cong \mathrm{Emb}([1], X) \coloneqq \{f : [1] \to X\},$$

where $[1] = \{1\}$ and $\mathrm{Emb}(A, X)$ is the space of embeddings of A into X.

By this way, we may extend the space X as follows: since X is a collection of embeddings of one-point, we now consider collections of embeddings of two points, three-points, and n points in general, which will be denoted by $\mathrm{Conf}_n(X)$ and called the *ordered n-configuration space* of X

$$\mathrm{Conf}_n(X) \coloneqq \mathrm{Emb}([n], X), \qquad [n] = \{1, 2, \ldots, n\}.$$

One can easily check that

Lemma 1.1. *The following holds:*

(1) *the ordered* 0*-configuration space* $\mathrm{Conf}_0(X)$ *is the singleton set.*

$$\mathrm{Conf}_0(X) = \{*\}$$

(2) *the ordered* 1*-configuration space* $\mathrm{Conf}_1(X)$ *is the space* X *itself.*

$$\mathrm{Conf}_1(X) \cong X$$

(3) *the ordered n-configuration space is the complement*

$$\mathrm{Conf}_n(x) \cong X^n \setminus \Delta = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \textit{ if } i \neq j\},$$

where Δ *is the* big diagonal *of* X^n.

For example, for the interval $X = (0,1)$,

$$\begin{aligned}
\mathrm{Conf}_1(X) &= X,\\
\mathrm{Conf}_2(X) &= \{(x_1,x_2) \in X^2 \mid x_1 \neq x_2\},\\
\mathrm{Conf}_3(X) &= \{(x_1,x_2,x_3) \in X^2 \mid x_1 \neq x_2 \neq x_3, x_1 \neq x_3\}.
\end{aligned}$$

Notice that $\mathrm{Conf}_n(X)$ is nothing but $n!$-copies of the open n-simplex. Indeed, for any X, there is an action of the symmetric group $\mathbf{S}_n$ by permuting coordinates. Namely, for each $\sigma \in \mathbf{S}_n$ and $\mathbf{x} = (x_1,\ldots,x_n) \in \mathrm{Conf}_n(X)$,

$$\sigma \cdot \mathbf{x} \coloneqq (x_{\sigma(1)},\ldots,x_{\sigma(n)}).$$

Then the group $\mathbf{S}_n$ acts on $\mathrm{Conf}_n(X)$ properly discontinuously so that the orbit space denoted by $B_n(X)$ admits the covering space $\mathrm{Conf}_n(X)$ with the deck transformation group $\mathbf{S}_n$ and is homeomorphic to the space of unordered distinct n points in X

$$B_n(X) = \mathrm{Conf}_n(X)/\mathbf{S}_n \cong \{\{x_1,\ldots,x_n\} \subset X \mid x_i \neq x_j \text{ if } i \neq j\}.$$

We call $B_n(X)$ the *(unordered) n-configuration space* of X and define the (indexed) union

$$B(X) \coloneqq \coprod_{n\geq 0} B_n(X)$$

called the *total (unordered) configuration space* of X.

The total configuration space has the following properties:

Proposition 1.2. *For each embedding $f : X \to Y$, there is an induced embedding*

$$B(f) : B(X) \to B(Y)$$

which makes B a functor

$$B : \mathcal{E}\mathsf{mb} \to \mathcal{E}\mathsf{mb},$$

where $\mathcal{E}\mathsf{mb}$ is the subcategory of $\mathcal{T}\mathsf{op}$ with embeddings as morphisms.

Proposition 1.3. *For two spaces X, Y, there is a canonical homeomorphism*

$$B\left(X \coprod Y\right) = B(X) \times B(Y)$$

Therefore, the functor B is indeed a functor between *monoidal categories*

$$B : \left(\mathcal{E}\mathsf{mb}, \coprod\right) \to (\mathcal{E}\mathsf{mb}, \times).$$

2. Braid equivalences

It is obvious that for two homeomorphic spaces X and Y, their total configuration spaces are also homeomorphic and *vice versa*.

$$X \cong Y \Longleftrightarrow B(X) \cong B(Y)$$

Remark 2.1. Remember that $B(X)$ contains X as a connected summand.

Then how about the homotopy equivalences? Obviously, X and Y are homotopy equivalent if so are total configuration spaces $B(X)$ and $B(Y)$.

$$B(X) \simeq B(Y) \Longrightarrow X \simeq Y.$$

However, the converse is not always true. For example, the disk $X = D^2$ and the point $Y = *$ are homotopy equivalent but for $n \geq 2$, $B_n(Y) = \varnothing$ while $B_n(X) \neq \varnothing$.

Definition 2.2 (Braid equivalence). We say that two spaces X and Y are *braid equivalent* denoted by

$$X \overset{B}{\simeq} Y$$

if their total configuration spaces are homotopy equivalent.

The above discussion implies that the braid equivalence is strictly finer than the homotopy equivalence.

For the comparison between homeomorphicity and braid equivalent, we have the following lemma.

Lemma 2.3. *Let $X = D^2 \vee_\partial I$ be the boundary join of the disk D^2 and an interval I and let $Y = D^2$. Then X and Y are braid equivalent.*

In particular, the homeomorphicity is strictly finer than the braid equivalence.

Hence the braid equivalence is exactly in between homeomorphicity and homotopy equivalence

$$X \cong Y \underset{\not\Longleftarrow}{\Longrightarrow} X \overset{B}{\simeq} Y \underset{\not\Longleftarrow}{\Longrightarrow} X \simeq Y$$

and it gives us the following natural question.

Question. How much can the braid equivalence separate spaces? Or, how strong are homotopy invariants of total configuration spaces?

Remark 2.4. The reason why we call the above equivalence the *braid* equivalence is as follows: any path γ in the n-configuration space $B_n(X)$ will give us a simultaneous motion of n-particles on X without collapsing. If we present this motion as a graph of γ in the space $X \times I$, then we can really see the *braiding* of n-strands in $X \times I$.

3. Homotopy invariants

In this section, we will review known results about homotopy invariants such as homotopy groups or homology groups for total configuration spaces.

Theorem 3.1 (Farley-Sabalka, Ko-Park,...). *The following holds:*

(1) For compact surfaces X and Y,

$$X \overset{B}{\simeq} Y \Longleftrightarrow \pi_1(B(X)) \cong \pi_1(B(Y)) \Longleftrightarrow X \cong Y$$

(2) For trees T_1 and T_2,

$$T_1 \overset{B}{\simeq} T_2 \Longleftrightarrow H^*(B(T_1)) \cong H^*(B(T_2)) \Longleftrightarrow T_1 \cong T_2$$

Theorem 3.2 (A.-Park, pub. in Topol. Appl.). *For any finite simplicial complex X, the homotopy invariants determines the following embeddabilities:*

(1) X embeds into a circle S^1:

$$X \subset S^1 \Longleftrightarrow \pi_1(B_n(X)) = \mathbb{Z} \quad \exists \text{ or } \forall n \geq 3$$

(2) X embeds into a surface Σ different from $S^2, \mathbb{R}P^2$:

$$X \subset \Sigma \Longleftrightarrow \pi_1(B_n(X)) \text{ is torsion-free} \quad \exists \text{ or } \forall n \geq 2.$$

(3) X embeds into the plane $\mathbb{R}^2$:

$$X \subset \mathbb{R}^2 \Longleftrightarrow H_1(B_n(X)) \text{ is torsion-free} \quad \exists \text{ or } \forall n \geq 2.$$

As seen above, the homotopy invariants give us geometric informations for the underlying space X. In particular, it separates all surfaces and trees. However, only a few information had been known for graphs.

4. Configuration spaces of graphs

In the study of configuration spaces, it tends to be harder for graphs rather than for surfaces or manifolds.

One may see the reason below. Imagine a path (or a motion) of n distinct points in X. What happens if two points approach to each other? If the space they belong to is large enough, then points can avoid each other and go their own ways. How about the case when two points encounter each other on a single log bridge? Can they still change their position? The answer depends on whether our space X has another bridge or there is a spot for one to give way to the other, which is a question about the global shape of the space. This interesting situation is due to that our space is assumed to be very restrictive, just like a graph, which is a one-dimensional singular space.

Indeed, studying configuration spaces of graphs is related with many real-life questions including robots in a factory, trains on the tracks, motion planning and so on. Rather deeply, one may find the relation between entanglements of particles over graphs, a braiding, and architectures for topological quantum computers.

Mathematically, these braiding can be captured by homology cycles.

The configuration space of one particle over the graph is nothing but the graph itself and so the only nontrivial motion is represented by a loop of the graph. Hence we are able to detect loops from the configuration space. For two particles over the graph, we have additional nontrivial motion, called a *star move* (or *star class*), that occurs at whereever the three-way junction exists.

(a) A loop move (b) A star move

FIGURE 1. One- and two-particle motions over graphs

These two moves are elementary moves and enough to describe *one-dimensional motions*. Two- or higher-dimensional motions may exist as well even though it is not easy to imagine them. One easiest example of such motions is to consider loop and star moves simultaneously and independently at various and disjoint positions.

4.1. Edge stabilizations and module structures. On the other hand, the configuration spaces of graphs have an action of edges, called a *stabilization* which adds a particle on the desired edge. This is never possible for non-graphs and probably no one believes that the stabilization is possible since the manipulation of configuration spaces is hardly continuous.

THEOREM 4.1 (A.–Drummond-Cole–Knudsen, pub. in Doc. Math., Geom. Topol.). *For a finite graph $\Gamma = (V, E)$, the following holds.*

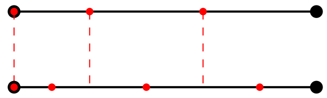


FIGURE 2. Stabilization on an edge by taking averages

(1) The toal configuration space $B(X)$ has $\mathbb{Z}[E]$-action and so the singular chain complex $C(B(X))$ is a $\mathbb{Z}[E]$-module.
(2) There exists a finitely generated double complex $S(\Gamma)$, called a Swiatkowski*'s complex, which is $\mathbb{Z}[E]$-module such that there is a quasi-equivalence*

$$S(\Gamma) \simeq H_*(C^{\mathsf{sing}}(B(X)))$$

as $\mathbb{Z}[E]$-modules.

4.2. Fomalities. Now we have a $\mathbb{Z}[E]$-module structure on homology group and we may ask its formality.

REMARK 4.2. Once it turns out to be formal, then the Künneth spectral sequence will collapse and so we may build up the homology group of larger graphs from those of smaller graphs by the cut-and-paste method.

THEOREM 4.3 (A.–Drummond-Cole–Knudsen, pub. in Doc. Math., Geom. Topol.). *As $\mathbb{Z}[E]$-modules, the homology group $H_*(B(X))$ is formal if and only if Γ is one of small graphs depicted in Figure 3.*

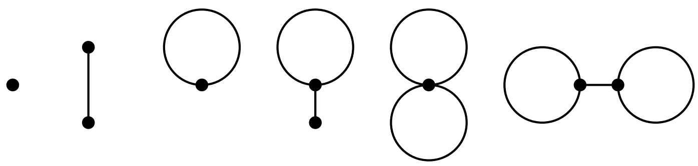


FIGURE 3. Complete list of edge-formal graphs

4.3. Betti number polynomials and growths. One another observation is that since E is finite and $S(\Gamma)$ is finitely generated, the sequence of i-th betti numbers will be eventually a polynomial of n. That is, for each i, there exists a polynomial $P_i^\Gamma(n)$ such that

$$P_i^\Gamma(n) = \operatorname{rank}(H_i(B_n(\Gamma))), \quad n \gg 1.$$

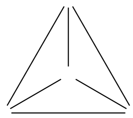
What we have found is the degree of $P_i^\Gamma(n)$ is indeed the combinatorial invariant of Γ, which is the maximal possible number of pieces after removing i-vertices from Γ.

THEOREM 4.4 (A.–Drummond-Cole–Knudsen, pub. in Geom. Topol.). *For $i \geq 2$,*

$$\deg P_i^\Gamma(n) = \Delta_i^\Gamma - 1,$$

where

$$\Delta_i^\Gamma = \max_{\substack{W \subset V \\ |W|=i}} |\pi_0(\Gamma \setminus W)|.$$



i	Δ_i^Γ	$\Delta_i^\Gamma - 1$	$\dim H_i B_k(\Gamma)$	valid for
0	1	0	1	all k
1	1	0	4	$k \geq 2$
2	2	1	$6k - 15$	$k \geq 3$
3	4	3	$4\binom{k-3}{3}$	all k
4	6	5	$\binom{k-3}{5}$	all k
≥ 5	$-\infty$	$-\infty$	0	all k

FIGURE 4. A verification of Theorem 4.4

5. Conclusion

The study of graphs via configuration space has turned out to be very useful. It gives us lots of information, most of which are distinguishable from those coming from combinatorics.

Although we have successfully found some nice invariants from the configuration spaces, we still believe that a lot more interesting aspects are there under the veil.

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PROPERTIES OF SOLITONS FOR THE INVERSE MEAN CURVATURE FLOW

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Abstract. Inverse mean curvature flow has been extensively studied as a geometric flow and for applications to establish several geometric inequalities. Analyzing special solutions of types of geometric flow is helpful to understand the flow itself and so does the inverse mean curvature flow. In this talk, we consider the homothetic and translating solitons for the inverse mean curvature flow as special solutions deformed by only homothety and translation under the flow, respectively. To be specific, we introduce several examples of the solitons and then, the incompleteness of any translating soliton and the homothetic solitons with some restricted homothetic ratio, namely, 0¡ C ¡ 1/n. Their area growth is also provided.

1. Preliminaries

Definition 1.1 (Inverse mean curvature flow). A smooth family of immersions $F : \Sigma \times [0,T) \to \mathbb{R}^{n+1}$, $0 < T \leq \infty$ is called a solution of the *inverse mean curvature flow* (IMCF) if F_t satisfies the equation

$$\frac{\partial}{\partial t} F_t(p) = -\frac{1}{H_t(p)} \nu_t(p),$$

for any $p \in \Sigma$ and $t \in [0,T)$, where H_t and ν_t are the mean curvature and inward unit normal vector field of F_t, respectively.

We shortly introduce two inequalities using IMCF: Gerhardt and Urbas (1990) proved independently that a closed, star-shaped, mean-convex Euclidean hypersurface evolves into a round sphere along IMCF after rescaling. The following quotient of a closed Euclidean hypersurface Σ

$$Q(\Sigma) = \frac{\int_\Sigma H dA}{|\Sigma|^{\frac{n-1}{n}}},$$

is monotone decreasing (P. Guan-J. Li, 2009) if we deform Σ by IMCF. Minkowski's inequality in arbitrary dimensions:

$$\frac{\int_\Sigma H dA}{|\Sigma|^{\frac{n-1}{n}}} \geq \frac{n|\mathbb{S}^n|}{|\mathbb{S}^n|^{\frac{n-1}{n}}} = n|\mathbb{S}^n|^{\frac{1}{n}}.$$

Let Σ be a closed surface in a Riemannian manifold (M^3, g). The Hawking mass of Σ is defined by

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi}\int_\Sigma H^2 dA\right).$$

Geroch showed that if M has $R_g > 0$, then $m_H(\Sigma_t)$ is monotone increasing under the IMCF. If we assume that M is asymptotically flat, then $m_H(\Sigma_t) \to m_{ADM}(M)$ as $t \to \infty$. By taking Σ_0 as the event horizon, Riemannian Penrose inequality is obtained as follows:

$$m_{ADM}(M) \geq m_H(\Sigma_0) = \sqrt{\frac{|\Sigma_0|}{16\pi}}.$$

However, a weak solution approach resolved by Huisken and Ilmanen is needed from $H \equiv 0$ at $t = 0$.

2. Solitons for the inverse mean curvature flow

2.1. Homothetic solitons.

Definition 2.1 (Homothetic soliton). The hypersurface Σ is called a homothetic soliton if there exists a nonzero constant C such that

$$\langle \overrightarrow{H}, x \rangle = -\frac{1}{C},$$

where $\overrightarrow{H}$ is the mean curvature vector of Σ.

The homothetic soliton has a solution form $\{\Sigma_t = e^{Ct}\Sigma\}$. If C is positive/negative, then Σ is an expender/shrinker. The minimal surface and the zero scalar curvature surfaces have the scaling invariant, and so does the homothetic soliton. Huisken and Ilmanen [6] proved existence of several rotationally symmetric homothetic solitons were introduced from a dynamical system using phase-plane analysis. Drugan, Lee and Wheeler [3] proved the followings:

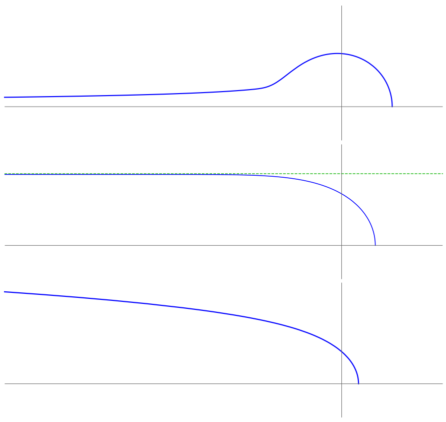


Figure 5. Self-expanding solutions of inverse mean curvature flow

(1) A closed homohtetic soliton for IMCF is a round hypersphere with $C = \frac{1}{n}$.
(2) They classified one-dimensional homothetic solitons.
(3) They constructed topological hypercylinder homothetic solitons that are *infinite bottles* for $C = \frac{1}{n-1}$.

Hui [5] proved the existence of a unique even solution $(x, y(x))$ of the rotationally symmetric homothetic soliton for $C > \frac{1}{n-1}$ was obtained. We consider an n-dimensional rotationally symmetric hypersurface for the IMCF, namely, $X : I \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$ parametrized by

$$X(s, \phi_1, \cdots, \phi_{n-1}) = (x(s)\Phi(\phi_1, \cdots, \phi_{n-1}), y(s)),$$

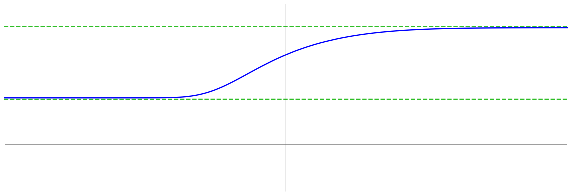


FIGURE 6. An infinite bottle.

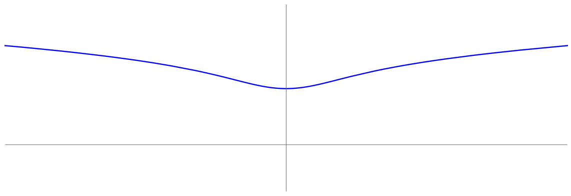


FIGURE 7. The even solution for $C > \frac{1}{n-1}$

where Φ is an orthogonal parametrization of the $(n-1)$-dimensional unit sphere. The profile curve $\gamma(s) = (x(s), y(s))$ on $\mathcal{Q} = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0\}$ satisfies the following ordinary differential equation:

$$-\frac{1}{C} + \frac{1}{x}(xy' - yx')(y'(n-1-xx'') + xx'y'') = 0, \tag{1}$$

for some nonzero constant C. We define a coordinate transformation used in [1] by taking a suitable parameter s $(-\infty < s < \infty)$ as follows:

$$\tan(a) = \frac{y'}{x'} \quad \text{and} \quad \tan(b) = \frac{y}{x}. \tag{2}$$

Changing the coordinates in the equation (1) and multiplying by $2C\cos(b)b'$, we obtain

$$Aa' + Bb' = 0,$$

where

$$A = 2C\cos(b)\sin^2(a-b),$$
$$B = (C(n-1)-2)\cos(b) - C(n-1)\cos(2a-b).$$

We define an associated vector field

$$V(a,b) = (V_1(a,b), V_2(a,b)) = (a', b') = (B, -A).$$

From the dynamical system, we can completely classify all solutions to be rotationally symmetric homothetic solitons for IMCF as follows:

THEOREM 2.2. *There are no complete proper homothetic solitons for* $0 < C < \frac{1}{n}$ *in* $\mathbb{R}^{n+1}$.

PROOF. Suppose that M is such homothetic soliton. We take the sphere centered the origin with sufficiently small enough radius r so that is disjointed with M. If the radius r increases, then there is a first touching point.

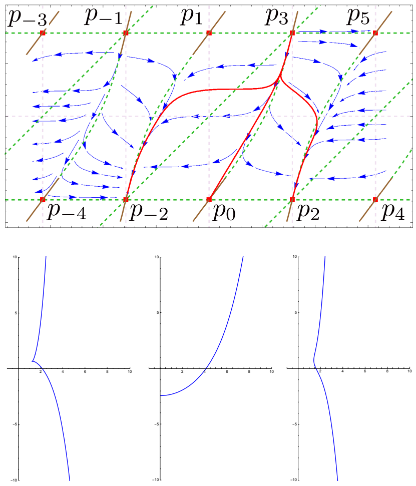


FIGURE 8. Case 1. $C > \frac{1}{n-1}$: Integral curves and profile curves for $n = 5$.

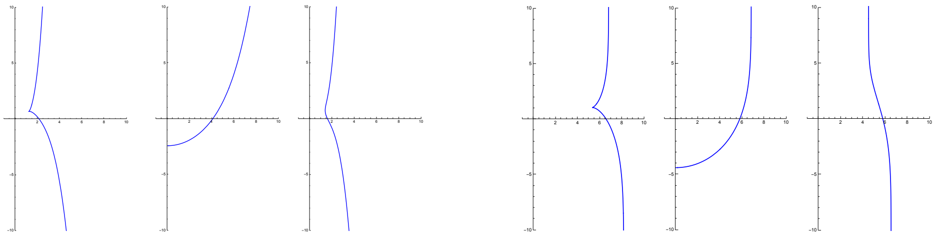


FIGURE 9. Case 1. $C > \frac{1}{n-1}$ and Case 2. $C = \frac{1}{n-1}$.

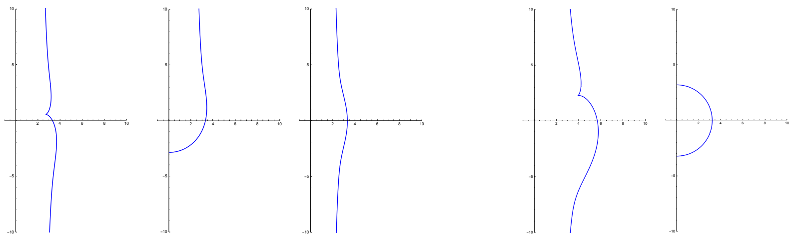
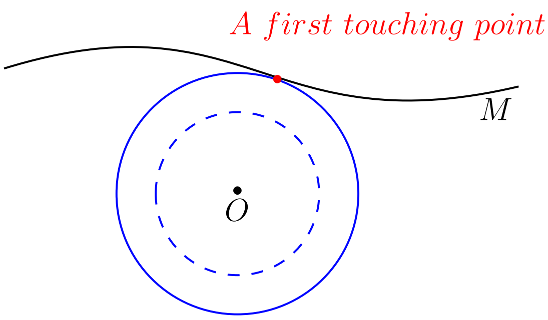


FIGURE 10. Case 3. $\frac{1}{n} < C < \frac{1}{n-1}$ and Case 4. $C = \frac{1}{n}$.



We define the function $f(x) = \|x\|^2$ on M so that

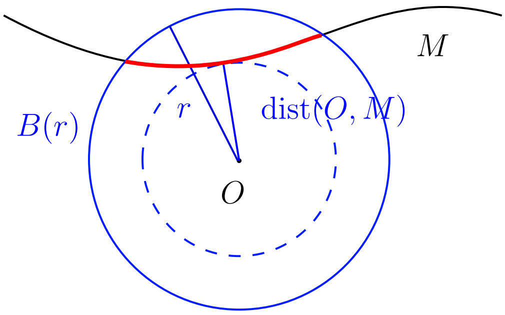
$$\triangle \|x\|^2 = 2\left(n - \frac{1}{C}\right) < 0.$$

By maximum principle, the minimum value of f does not on the interior of M. The function f has the minimum value at p, which is a contradiction. $\square$

THEOREM 2.3. *Let M be an n-dimensional complete hypersurface M. Suppose that M is a homothetic soliton with $n - \frac{1}{C} = \alpha \geq 0$. Then, there exists a positive constant $c = c(\alpha)$ such that*

$$cr^{\alpha} \leq \mathrm{Vol}(B(r) \cap M),$$

where $B(r)$ is the $(n+1)$-dimensional ball centered at the origin of the radius $r > r_1$ for some $r_1 > \mathrm{dist}(0, M)$.



PROOF. By direct computation, we have

$$\triangle \|x\|^2 = 2\left(n - \frac{1}{C}\right) = 2\alpha \geq 0. \tag{3}$$

By the divergence theorem, the equation (3) implies that

$$2\alpha \mathrm{Vol}(M \cap B(r)) = \int_{M \cap B(r)} \triangle \|x\|^2 = \int_{\partial(M \cap B(r))} \left\langle \nabla \|x\|^2, \eta \right\rangle,$$

where η is the outward unit normal vector of $\partial(M \cap B(r))$. By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\int_{\partial(M\cap B(r))} \left\langle \nabla \|x\|^2, \eta \right\rangle &= \int_{\partial(M\cap B(r))} 2\left\langle x^{\top}, \eta \right\rangle = \int_{\partial(M\cap B(r))} 2\langle x, \eta\rangle \\ &\leq \int_{\partial(M\cap B(r))} 2\|x\| = 2r\mathrm{Vol}(\partial(M \cap B(r))).\end{aligned}$$

From the coarea formula, we obtain

$$\alpha \mathrm{Vol}(M \cap B(r)) \leq r\mathrm{Vol}(\partial(M \cap B(r))) \leq r\left(\frac{d}{dr}\mathrm{Vol}(M \cap B(r))\right).$$

That implies that

$$0 \leq \frac{\alpha}{r} \leq \frac{1}{\mathrm{Vol}(M \cap B(r))}\left(\frac{d}{dr}\mathrm{Vol}(M \cap B(r))\right). \tag{4}$$

Integrating to both sides of the inequality, we have

$$cr^{\alpha} \leq \mathrm{Vol}(M \cap B(r))).$$

where c is a constant depending on α. If $\alpha > 0$, then $\mathrm{Vol}(M)$ diverges by letting $r \to \infty$. $\square$

2.2. Translating soliton for IMCF.

Definition 2.4 (Translating soliton for IMCF). The hypersurface Σ is called a translating soliton for IMCF if there exists a non-zero constant vector field v satisfying

$$\langle \overrightarrow{H}, \mathrm{v} \rangle = -1,$$

where v is the direction of translation under the IMCF and it is called the translating direction.

The translating solitons are solutions $\{\Sigma_t = \Sigma + \mathrm{v}t\}$ satisfying the IMCF equation. We may assume that v is e_{n+1} up to rescaling and rotation in $\mathbb{R}^{n+1}$. Drugan, Lee and Wheeler [3] proved the followings:

(1) The only one-dimentional translating soliton is a cycloid.
(2) They introduced the one-parameter family of translating solitons as a combination of rotation and scaling of the cycloid cylinder.

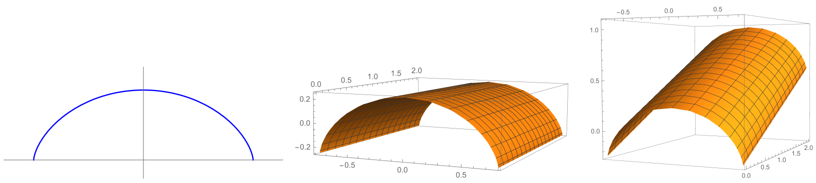


Figure 11. The cycloid and cycloid cylinders.

Theorem 2.5 (Kim and Pyo [7]). *The only ruled translating solitons for IMCF are cycloid cylinders.*

We consider an n-dimensional rotationally symmetric hypersurface, namely, $X : I \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$ parametrized by

$$X(s, \phi_1, \cdots, \phi_{n-1}) = (x(s)\Phi(\phi_1, \cdots, \phi_{n-1}), y(s)),$$

where Φ is an orthogonal parametrization of the $(n-1)$-dimensional unit sphere.

The profile curve $\gamma(s) = (x(s), y(s))$ on $\mathcal{Q} = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$ satisfies the following equivalent ordinary differential equation:

$$(n-1)x'y' + x(x'(y''x' - y'x'') + 1) = 0.$$

We define an appropriate coordinate transformation as follows:

$$\tan(a) = \frac{y'}{x'}, \quad \tan(b) = x. \tag{5}$$

Using the coordinate transformation and multiplying it by $\cos^4(b)b'$, we obtain

$$Aa' - Bb' = 0,$$

where

$$\begin{aligned} A &= \cos^2(a)\sin^2(b)\cos^2(b)\left(h^2\cos^2(b) + \sin^2(b)\right), \\ B &= -\sin(b)(\sin(a)\cos(a)\cos(b) + \sin(b))\left(2h^2\cos^2(a)\cos^2(b) + \sin^2(b)\right) \\ &\quad - h^4\cos^4(a)\cos^4(b). \end{aligned}$$

We define an associated vector field

$$V(a, b) = (V_1(a, b), V_2(a, b)) = (a', b') = (B, A).$$

From the dynamical system, we can completely classify all solutions to be rotationally symmetric translating solitons for IMCF as follows:

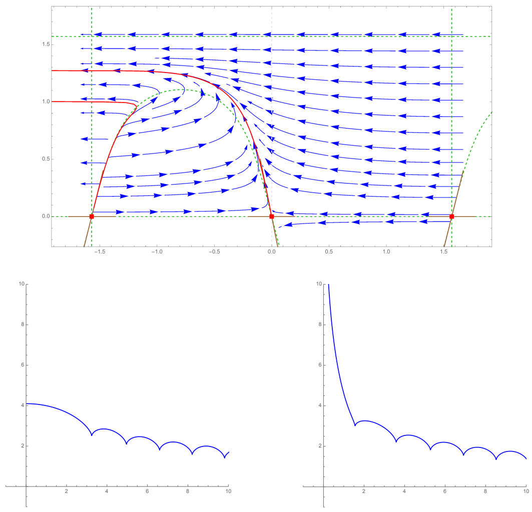


Figure 12. Rotationally symmetric translating solitons.

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REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS SATISFIES GENERALIZED KILIING CONDITION OF SYMMETRIC OPERATORS

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In 20th century, classifications with certain geometric problems for real hypersurfaces in spaces of complex space form or quaternionic space form were main research subjects in the field of differential geometry ([16], [17], [18]). Recently, many kinds of geometric problems have been considered on the classification of real hypersurfaces in the complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2{\cdot}U_m)$ or complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$ (see [1], [2], [4], [20], [22], [23], [26], [28], and [29]).

More generally, real hypersurfaces in Hermitian symmetric spaces with shape operator have been investigated by Berndt and Suh [1], [2], [3] and [4], Martinez and Pérez [16], Pérez and Suh [19] and [21].

In this paper, we want to apply this notion of Killing tensor to the structure Jacobi operator $\mathbb{R}_\xi$ which is a symmetric tensor field of type (0,2) on a Riemannian manifold (M, g). It is defined by

$$\mathbb{R}_\xi(X, Y) = g(R_\xi X, Y)$$

for the structure Jacobi operator R_ξ of type (1,1) and any vector fields X and Y on (M, g). The symmetric structure Jacobi operator $\mathbb{R}_\xi$ of type (0,2) on a Riemannian manifold M is said to be *generalized Killing* if it satisfies

$$(\nabla_X \mathbb{R}_\xi)(X, X) = g\left((\nabla_X R_\xi)X, X\right) = 0$$

for any vector field $X \in T_zM$, $z \in M$. This equation is equivalent to cyclic parallel structure Jacobi operator

$$\mathfrak{S}_{X,Y,Z}\left(\nabla_X \mathbb{R}_\xi\right)(Y, Z) = \mathfrak{S}_{X,Y,Z}\, g\left((\nabla_X R_\xi)Y, Z\right) = 0$$

for any X, Y and $Z \in T_zM$, $z \in M$, because of polarization, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to the vector fields X, Y and Z. That is, the condition of generalized Killing structure Jacobi operator R_ξ of M implies

$$\begin{aligned} & g\left((\nabla_X R_\xi)X, X\right) = 0 \\ \Longleftrightarrow\ & g\left((\nabla_X R_\xi)Y, Z\right) + g\left((\nabla_Y R_\xi)Z, X\right) + g\left((\nabla_Z R_\xi)X, Y\right) = 0 \\ \Longleftrightarrow\ & \mathfrak{S}_{X,Y,Z}\, g\left((\nabla_X R_\xi)Y, Z\right) = 0. \end{aligned} \tag{6}$$

Here, we can give the geometric meaning of the generalized Killing structure Jacobi operator as follows: When we consider a geodesic γ with initial conditions such that $\gamma(0) = z$ and $\dot\gamma(0) = X$. Then the structure Jacobi curvature $\mathbb{R}_\xi(\dot\gamma, \dot\gamma) = g(R_\xi\dot\gamma, \dot\gamma)$ is constant along the geodesic γ of the vector field X (see Semmelmann [24]).

Now, in this paper we consider a real hypersurface M in complex hyperbolic two-plane Grassmanians $SU_{2,m}/S\,(U_2 \cdot U_m)$ with generalized Killing structure Jacobi operator.

From such a view point, in a direction of generalized Killing structure Jacobi operator for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ we gave an important result. In fact, recently, for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Killing structure Jacobi operator Lee, Suh, and Woo [15] gave a classification theorem as follows:

Theorem B. Let M be a Hopf real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the structure Jacobi operator R_ξ of M is generalized Killing if and only if M is locally congruent to an open part of a tube of $r = \frac{\pi}{4\sqrt{2}}$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

On the other hand, a real hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$ is said to be Hopf if the shape operator A of M satisfies $A\xi = \alpha\xi$, $\alpha = g(A\xi, \xi)$, for the Reeb vector feld $\xi = -JN$, where N denotes a unit normal vector field on M.

Motivated by this result, it is natural to consider a generalized Killing structure Jacobi operator for real hypersurfaces M in $SU_{2,m}/S(U_2 \cdot U_m)$. In this paper, we want to consider a new notion of the generalized Killing structure Jacobi operator R_ξ of M in $SU_{2,m}/S(U_2 \cdot U_m)$ defined by

$$\mathfrak{S}_{X,Y,Z}\, g\left(\left(\nabla_X R_\xi\right) Y, Z\right) = 0 \tag{*}$$

for any tangent vector fields X, Y, and Z on M. Then we can assert the following

Main Theorem. There does not exist a connected Hopf real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with generalized Killing structure Jacobi operator.

As mentioned above, the notion of Killing symmetric tensor is a kind of generalized notion of parallelism and can be regarded as the symmetric tensor of a Riemannian manifold. It means that if the symmetric tensor T is parallel, that is, $\nabla T = 0$, then T can be generalized Killing. If we apply such a relation to the structure Jacobi operator R_ξ for a real hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, we can give the following result from our Main Theorem.

Corollary. There does not exist a connected Hopf real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with parallel structure Jacobi operator.

1. Key lemma

Let $SU_{2,m}/S(U_2 \cdot U_m)$ and M be a complex hyperbolic two-plane Grassmannian and its Hopf real hypersurface such that $A\xi = \alpha\xi$, respectively. Hereafter, unless otherwise stated, we consider that X and Y are any tangent vector fields on M. The structure Jacobi operator R_ξ of M in $SU_{2,m}/S(U_2 \cdot U_m)$ is given by

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= -\frac{1}{2}\Big[X - \eta(X)\xi - \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu\Big\} \\ &\qquad - \sum_{\nu=1}^{3}\Big\{3g(\phi_\nu X,\xi)\phi_\nu K + \eta_\nu(\xi)\phi_\nu PX\Big\}\Big] + \alpha AX - \eta(AX)A\xi, \end{aligned}$$

where the function α is defined by $\alpha = g(A\xi, \xi)$ and said to be the *Reeb function* on M (see [30]).

We can rearrange the generalized Killing structure Jacobi operator as follows:

$$
\begin{aligned}
0 =&g(\phi AX,Y)\xi+\eta(Y)\phi AX\\
&+2\eta((\nabla_X A)\xi)AY+2\alpha(\nabla_X A)Y-2\alpha\eta((\nabla_X A)Y)\xi-2\alpha g(AY,\phi AX)\xi\\
&-2\alpha\eta(Y)(\nabla_X A)\xi-2\alpha\eta(Y)A\phi AX+g(\phi AY,X)\xi+\eta(X)\phi AY\\
&+2\eta((\nabla_Y A)\xi)AX+2\alpha(\nabla_Y A)X-2\alpha\eta((\nabla_Y A)X)\xi-2\alpha g(AX,\phi AY)\xi\\
&-2\alpha\eta(X)(\nabla_Y A)\xi-2\alpha\eta(X)A\phi AY-\eta(Y)A\phi X-\eta(X)A\phi Y\\
&+2(\xi\alpha)g(AX,Y)\xi-4(\xi\alpha)g(AX,Y)\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi\xi_\nu+2\alpha(\nabla_X A)Y\\
&-\alpha\eta(X)\phi Y-\alpha g(\phi X,Y)\xi-2\alpha\eta(Y)\phi X-4\alpha(\xi\alpha)\eta(X)\eta(Y)\xi\\
&+2\alpha\eta(Y)A\phi AX+2\alpha\eta(X)A\phi AY\\
&+\sum_{\nu=1}^{3}\big[g(\phi_\nu AX,Y)\xi_\nu-2\eta(Y)\eta_\nu(\phi AX)\xi_\nu+\eta_\nu(Y)\phi_\nu AX\\
&\qquad+3g(\phi_\nu AX,\phi Y)\phi_\nu\xi+3\eta(Y)\eta_\nu(AX)\phi_\nu\xi+3\eta_\nu(\phi Y)\phi_\nu\phi AX\\
&\qquad-3\alpha\eta_\nu(\phi Y)\eta(X)\xi_\nu+4\eta_\nu(\xi)\eta_\nu(\phi Y)AX\\
&\qquad-4\eta_\nu(\xi)g(AX,Y)\phi_\nu\xi+2\eta_\nu(\phi AX)\phi_\nu\phi Y\big]\\
&+\sum_{\nu=1}^{3}\big[g(\phi_\nu AY,X)\xi_\nu-2\eta(X)\eta_\nu(\phi AY)\xi_\nu+\eta_\nu(X)\phi_\nu AY\\
&\qquad+3g(\phi_\nu AY,\phi X)\phi_\nu\xi+3\eta(X)\eta_\nu(AY)\phi_\nu\xi+3\eta_\nu(\phi X)\phi_\nu\phi AY\\
&\qquad-3\alpha\eta_\nu(\phi X)\eta(Y)\xi_\nu+4\eta_\nu(\xi)\eta_\nu(\phi X)AY\\
&\qquad-4\eta_\nu(\xi)g(AY,X)\phi_\nu\xi+2\eta_\nu(\phi AY)\phi_\nu\phi X\big]\\
&+\sum_{\nu=1}^{3}\big[-\eta_\nu(Y)A\phi_\nu X+2\eta(X)\eta_\nu(Y)A\phi\xi_\nu-\eta_\nu(X)A\phi_\nu Y\\
&\qquad+3\eta_\nu(\phi Y)A\phi_\nu\phi X-3\eta(X)\eta_\nu(\phi Y)A\xi_\nu+3\eta_\nu(\phi X)A\phi\phi_\nu Y\\
&\qquad-3\alpha\eta_\nu(\phi X)\eta_\nu(Y)\xi+4\eta_\nu(\xi)\eta_\nu(\phi X)AY\\
&\qquad+4\eta_\nu(\xi)\eta_\nu(\phi Y)AX-2g(\phi_\nu\phi X,Y)A\phi\xi_\nu\big]\\
&+\alpha\sum_{\nu=1}^{3}\big[-\eta_\nu(X)\phi_\nu Y-g(\phi_\nu X,Y)\xi_\nu-2\eta_\nu(Y)\phi_\nu X\\
&\qquad+\eta_\nu(\phi X)\phi\phi_\nu Y+g(\phi_\nu\phi X,Y)\phi\xi_\nu+4\eta(X)\eta(Y)\eta_\nu(\xi)\phi\xi_\nu\\
&\qquad-\eta(X)\eta_\nu(Y)\phi\xi_\nu-\eta_\nu(\phi X)\eta_\nu(Y)\xi-4\eta(X)\eta_\nu(\xi)\eta_\nu(\phi Y)\xi\big].
\end{aligned}
\tag{4.1}
$$

Then by virtue of (4.1), we can prove the following:

LEMMA 1.1. *Let M be a Hopf real hypersurface in the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2\cdot U_m)$, $m\geq 3$, with generalized Killing structure Jacobi operator. Then the Reeb vector field ξ belongs to either the maximal quaternionic subbundle $\mathcal{Q}$ or its orthogonal complement $\mathcal{Q}^\perp$.*

2. The Reeb vector field $\xi\in\mathcal{Q}^\perp$

Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2\cdot U_m)$ with generalized Killing structure Jacobi operator. Then by Lemma 1.1 we shall make an investigation into two cases depending on ξ belongs to either distribution $\mathcal{Q}^\perp$ or distribution $\mathcal{Q}$, respectively. So, in this section let us consider the case $\xi \in \mathcal{Q}^{\perp}$ (i.e., $JN \in \mathfrak{J}N$ where N is a unit normal vector field on M in $SU_{2,m}/S(U_2 \cdot U_m)$). Since $\mathcal{Q}^{\perp} = \mathrm{span}\{\xi_1, \xi_2, \xi_3\}$, we may put $\xi = \xi_1$. By using this equation we obtain:

LEMMA 2.1. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$ and $\xi \in \mathcal{Q}^{\perp}$, then*
(i) $\phi AX = 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX$ *and*
(ii) $A\phi X = 2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 + A\phi_1 X$.

From now on, by using this lemma, let us consider our classification problem with respect to the notion of generalized Killing structure Jacobi operator of a real hypersurface with $\xi \in \mathcal{Q}^{\perp}$ in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$.

We have

$$\begin{aligned}2(\nabla_\xi R_\xi)X = {} & 2(\xi\alpha)AX + 2\alpha(\nabla_\xi A)X - 4\alpha(\xi\alpha)\eta(X)\xi \\ & - 4\alpha\sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi_\nu\xi\} \\ & + 4\alpha\sum_{\nu=1}^{3}\{\eta_\nu(\xi)\eta_\nu(\phi X)\xi - \eta_\nu(\xi)\eta(X)\phi_\nu\xi\}\end{aligned} \tag{5.1}$$

for any tangent vector field X on M.

On the other hand, we also have

$$\begin{aligned}2(\nabla_X R_\xi)\xi = {} & \phi AX - 2\alpha A\phi AX \\ & - \sum_{\nu=1}^{3}\{g(\phi_\nu AX, \xi)\xi_\nu - \eta_\nu(\xi)\phi_\nu AX\} \\ & + \sum_{\nu=1}^{3}\{3\eta_\nu(AX)\phi_\nu\xi - 8\eta_\nu(\xi)g(AX,\xi)\phi_\nu\xi\}.\end{aligned} \tag{5.2}$$

By using these equations, we assert:

LEMMA 2.2. *Let M be a real hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$ with generalized Killing structure Jacobi operator. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^{\perp}$, then the shape operator A commutes with the structure operator ϕ, that is, $A\phi = \phi A$.*

In the remained part of this section, by using Proposition A let us check whether the structure Jacobi operator R_ξ on a real hypersurface M_A of type $\mathcal{T}_A^*$ (or $\mathcal{H}_A^*$, resp.) satisfies generalized Killing condition. In order to do this, we assume that the structure Jacobi operator R_ξ of M_A is generalized Killing.

Then (4.1) can be rewritten as

$$\begin{aligned}0 = {} & (2+2\alpha^2)\phi AX - (2+2\alpha^2)A\phi X \\ & - \alpha\phi X + \alpha\phi_1 X + 2\alpha\eta_2(X)\xi_3 - 2\alpha\eta_3(X)\xi_2\end{aligned} \tag{5.3}$$

for any tangent vector field X on $T_z M_A$, $z \in M_A$.

Putting $X = \xi_2 \in T_\beta$ in (5.3) gives

$$4\alpha\xi_3 = 0. \tag{5.4}$$

Bearing in mind of Proposition A, we know that the Reeb function α is non-vanishing. From this fact, (5.4) gives $\xi_3 = 0$, which gives a contradiction.

3. The Reeb vector field $\xi \in \mathcal{Q}$

Due to Lemma 1.1, let us suppose that $\xi \in \mathcal{Q}$ (i.e., $JN \perp \mathfrak{J}N$) in this section. Related to this condition, Suh [25] proved:

Theorem C. Let M be a Hopf hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with the Reeb vector field belonging to the maximal quaternionic subbundle $\mathcal{Q}$. Then one of the following statements holds

($\mathcal{T}_B^*$) M is an open part of a tube around a totally geodesic $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2U_{2n})$, $m = 2n$,

($\mathcal{H}_B^*$) M is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JN \perp \mathfrak{J}N$, or

($\mathcal{E}$) The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$.

By virtue of this result, we assert that *a real hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$ satisfying the hypotheses in our main theorem is locally congruent to an open part of one of the model spaces mentioned in above Theorem* C. Hereafter, unless otherwise stated, such real hypersurfaces of type of $\mathcal{T}_B^*$, $\mathcal{H}_B^*$, and $\mathcal{E}$ in $SU_{2,m}/S(U_2 \cdot U_m)$ are denoted by M_B.

If we put $X = \xi_1 \in T_\beta$ in (4.1) and take the inner product with $\phi_1\xi$, then we have

$$4\beta(1 + \alpha^2) = 0. \tag{6.1}$$

On the other hand, putting $X = \phi_1\xi \in T_\gamma$ in (**??**) yields

$$\beta\xi_1 + 3\alpha\xi_1 - 3(3\alpha + \beta)\xi_1 = 0,$$

where we have used $A\phi_1\xi = 0$ and $\phi^2\xi_1 = -\xi_1$. Since ξ_1 is unit, this implies $\beta = -3\alpha$. Substituting this fact into (6.1) gives

$$-12\alpha(1 + \alpha^2) = 0.$$

Since it is known that the Reeb function α in Proposition B is non-vanishing, this implies a contradiction.

For a model space of $\mathcal{T}_B^*$, it is known that the principal curvatures α and β are given by $\alpha = \tanh(\sqrt{2}r)$ and $\beta = \coth(\sqrt{2}r)$. Substituting these ones into (6.1), we have

$$\tanh^2(\sqrt{2}r) + 1 = 0, \tag{6.2}$$

which gives us a contradiction.

Also in the case of real hypersurfaces M_B of type $\mathcal{H}_B^*$ or $\mathcal{E}$, since $\alpha = \sqrt{2}$ and $\beta = \sqrt{2}$, (6.1) becomes $0 = 2\sqrt{2}$. Thus we also have a contradiction.

This shows that real hypersurfaces M_B of types $\mathcal{T}_B^*$, $\mathcal{H}_B*$ or $\mathcal{E}$ cannot satisfy the condition of generalized Killing structure Jacobi operator, and therefore our Main Theorem for the case $\xi \in \mathcal{Q}$ can not be occurred.

Summing up these observations, we assert that the structure Jacobi operator R_ξ of real hypersurfaces of five types model spaces $\mathcal{T}_A^*$, $\mathcal{H}_A^*$, $\mathcal{T}_B^*$, $\mathcal{H}_B^*$ or $\mathcal{E}$ in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, does not satisfy such a notion of generalized Killing.

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LEGENDRIANS, LAGRANGIAN FILLINGS, AND CLUSTER STRUCTURES

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Abstract. The basic concepts relating Legendrians, and Lagrangians will be introduced. After that we briefly review known Legendrian invariants in several context, and discuss how cluster structure appears when we consider the space of Lagrangian fillings of a given Legendrian.

1. Introduction and preliminaries

Let us start with the basic notions of symplectic and contact geometry. A *symplectic manifold* (W, ω) consists of

$$\begin{cases} W \text{ is a } 2n\text{-dimensional manifold;} \\ \omega \in \Omega^2(W) \text{ satisfying } dw = 0, \omega^{\wedge n} \text{ is nowhere vanishing.} \end{cases}$$

In the symplectic manifold there is an important class of submanifold so called *Lagrangian submanifold*. More precisely, L is a Lagrangian submanifold in a symplectic manifold (W, ω) if and only if

$$\begin{cases} L \text{ is an } n\text{-dimensional submanifold;} \\ \omega|_{TL} \equiv 0. \end{cases}$$

Example 1.1. The main example of symplectic manifold is the cotangent bundle T^*M of a n-dimensional manifold M equipped with the standard symplectic form $\omega_{\text{std}} = \sum_{i=1}^n dp_i \wedge dq_i$, where $\{q_i\}_{i=1}^n$ are coordinates for the base manifold M and the $\{p_i\}_{i=1}^n$ for the fiber direction.

Typical examples of Lagrangian submanifolds in $(T^*M, \omega_{\text{std}})$ are the zero section o_M and the cofiber T_p^*M at a fixed point $p \in M$.

There is an odd-dimensional analogue of symplectic manifolds, so called, a contact manifold. A contact manifold $(Y, \ker \alpha)$ consist of

$$\begin{cases} Y \text{ is a } (2n-1)\text{-dimensional manifold;} \\ \alpha \in \Omega^1(Y) \text{ satisfying that } \alpha \wedge d\alpha^{\wedge n-1} \text{ is nowhere vanishing.} \end{cases}$$

Also, there is an analogous concept of Lagrangians in symplectic manifold, named, Legendrian submanifold. A submanifold Λ in a contact manifold $(Y, \ker \alpha)$ is called Legendrian if

$$\begin{cases} \Lambda \text{ is } (n-1)\text{-dimensional;} \\ \alpha|_{T\Lambda} \equiv 0. \end{cases}$$

Example 1.2. The contact and Legendrian manifolds naturally appear at the boundary of symplectic and Lagrangian manifolds, respectively. The unit cotangent bundle ST^*M with the canonical one-form $\lambda_{\text{can}} = \sum_{i=1}^n p_i dq_i$ gives a contact manifold. The unit co-sphere bundle ST_p^*M becomes a Legendrian submanifold in $(ST^*M, \xi = \ker \lambda)$.

2. Legendrian knots in $(\mathbb{R}^3, \xi_{\text{std}})$ and their invanriants

Darboux's theorem in contact geometry says that every contact manifold $(Y, \ker\alpha)$ locally contactomorphic to $(\mathbb{R}^{2n+1}, \ker(dz - \sum_{i=1}^n y_i dx_i))$. Especially when the contact manifold is three-dimensional, we have the standard local model

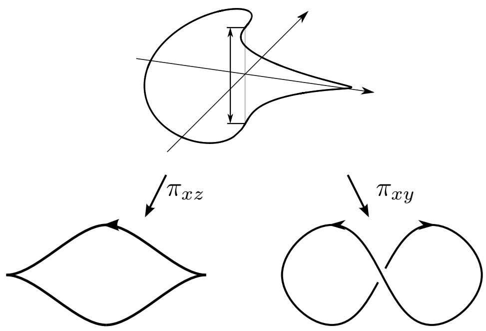
$$\left(\mathbb{R}^3, \xi_{\text{std}} = \ker(dz - ydx)\right).$$

In that standard space a one-dimensional Legendrian, i.e., *a Legendrian knot* $\Lambda : S^1 \to (\mathbb{R}^3, \xi_{\text{std}})$ satisfies

$$y(t) = \frac{\dot{z}(t)}{\dot{x}(t)}.$$

By the Legendrian condition it is enough to know two coordinates among three coordinates. There are two famous and meaningful projections, the front and Lagrangian projection:

$$\pi_{\text{F}} : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, z);$$
$$\pi_{\text{L}} : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, y).$$



We are interested in equivalence classes of Legendrian knots under Legendrian isotopy, which means smooth isotopy through Legendrian knots. This Legendrian isotopy can be interpreted as Reidemeister moves I, II, and, III in the front projection as depicted in Figure 13.

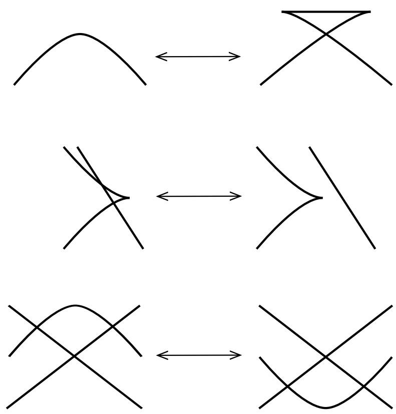


Figure 13. Reidemeister moves in the front projection

Here is the list of some Legendrian invariants:

(1) knot type;
(2) Thurston-Bennequin number;
(3) rotation number;
(4) ruling invariant;

(5) Legendrian contact algebra;
(6) augmentation category;
(7) constructible sheaf theoretic invariant;

Among the items (1), (2), and (3) are called *classical invariants* which can be determined by topology and contact homotopy data. Especially (2) and (3) can be computed in a combinatorial way in terms of the front projection:

$$tb(\Lambda) = \#\{\times, \times\} - \#\{\times, \times, \succ\};$$
$$rot(\Lambda) = \frac{1}{2}\left(\#\{\prec, \succ\} - \#\{\prec, \succ\}\right).$$

Briefly speaking, ruling invariant comes from a family of functions

$$\{F_x : \mathbb{R}^N \to \mathbb{R}_z\}_{x \in \mathbb{R}}$$

whose critical locus recovers the front projection of Legendrian knot. The above family of functions is called a *generating family*. There is a combinatorial way to encode the paring of the critical points for each F_x in a consistent way with respect to the x-coordinate, which is nothing but the *ruling*. A certain weighted counting of the ruling gives a polynomial invariant.

On the other hand, there is an alternative way of extracting Legendrian invariant by using J-holomorphic curve respecting a symplectic-Lagrangain pair

$$(\mathbb{R}^3 \times \mathbb{R}, d(e^t\lambda), \Lambda \times \mathbb{R}).$$

As a result, we obtain a differential graded algebra, whose generators are integral curves from the Legendrian to itself following the canonical vector field obtain from the contact 1-from.

The remaining items (6) and (7) is deeply related to each other and their (geometric) motivation is to consider the space of Lagrangian fillings of a Legendrian knot.

Let us depict an example of Lagrangian filling in $(\mathbb{R}^3 \times \mathbb{R}, d(e^t\lambda))$ especially for the case of Legendrian trefoil of maximal Thurston-Bennequin number as follows:

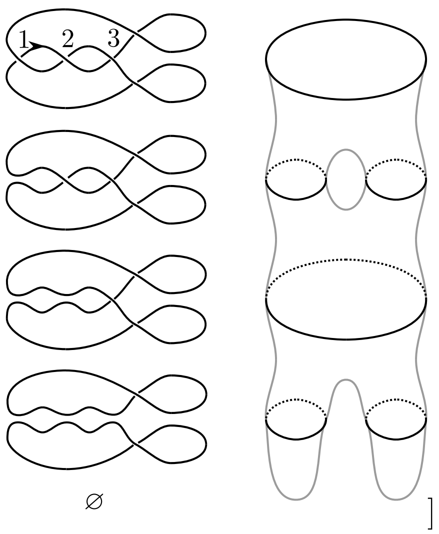
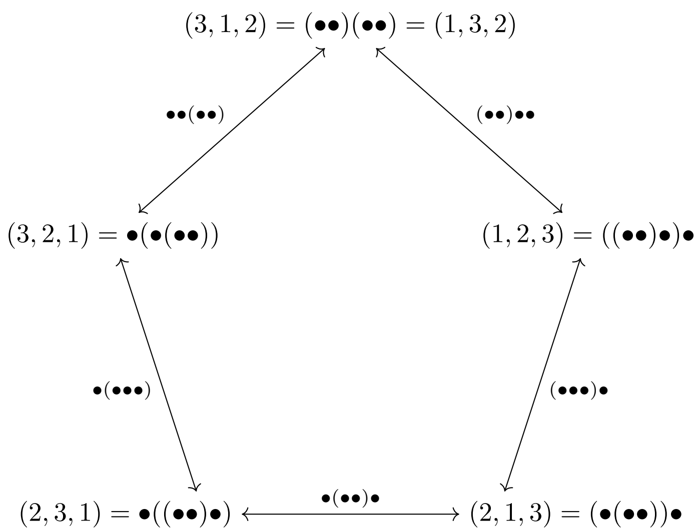


FIGURE 14. An exact Lagrangian filling for the Legendrian trefoil

Topological type of the filling is a torus with one puncture. If we restrict the topological type of the filling there are several inequivalent (exact) Lagrangian fillings up to Hamiltonian isotopy. This can be distinguished by using J-holomorphic curve theory, and constructible sheaf theory, independently. In the Legendrian trefoil case, there are at least five distinct fillings by considering different sequence of resolution of labelled crossings 1, 2, and 3 in Figure 14.

More interesting point is the relation and structure of those Lagrangian fillings as follows:



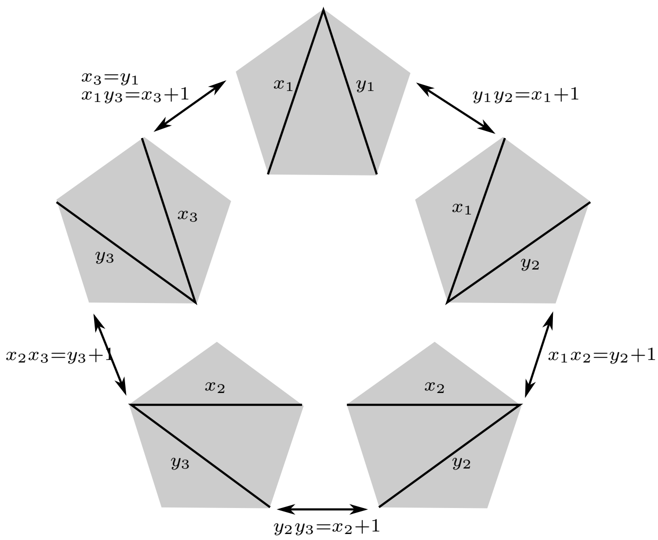


Figure 15. Cluster structure of A_2 type

Surprisingly, this structure is identical to the cluster structure of type A_2 in Figure 15. This phenomena is already extended to $(2, n)$-torus links and the cluster structure of type A_{n-1}. On the other hand, the cluster structure is deeply related to the theory of (plane curve) singularities as well as Dynkin diagrams. A conjecture says that this phenomenon also hold for the case of singularities of ADE-type (as well as BCFG-type).

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RATIONAL CURVES IN DEL-PEZZO VARIETIES

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Abstract. By the classification of smooth Fano 3-folds, the varieties $\mathbb{P}^3$, Q_3, V_5 and V_{22} are all of Fano 3-folds X with $\mathrm{Pic}(X) = \mathbb{Z}$ and $\mathrm{H}^3(X) = 0$. We review the geometry of moduli spaces of rational curves in the Fano variety with degree $d \leq 3$.

1. Introduction

Rational curves in Fano varieties have played useful roles in algebraic geometry as in the works of Clemens-Griffiths, Iskovski, Beauville-Donagai, Lehn-Lehn-Sorger-van Straten, Takkagi-Zucconi and Iliev-Manivel. Understanding the birational geometry of the moduli spaces of rational curves in a Fano variety may lead us to interesting examples of new varieties or may reveal some internal structure of Fano varieties ([CS09]). The purpose of this paper review the konwn result of the geometry of moduli spaces of rational curves of degree ≤ 3 in Fano 3-folds

$$\mathbb{P}^3, Q_3, V_5, \text{ and } V_{22}$$

Note taht all of varieties are rigid except V_{22}. The moduli of V_{22}'s is six-dimensional.

Let us introduce the well-known compactifications of rational curves space. Let $X \subset \mathbb{P}^r$ be a smooth projective variety with fixed ample line bundle $\mathcal{O}_X(1)$. Let $R_d(X)$ be the space of rational curves of fixed embedded degree $d \geq 1$.

• **Hilbert compactification**: Grothendieck's general construction gives us the Hilbert scheme $Hilb^{dt+1}(X)$ of closed subschemes of X with Hilbert polynomial $h(t) = dt+1$ as a closed subscheme of $Hilb^{dt+1}(\mathbb{P}^r)$. The closure $\mathbf{H}_d(X)$ of $R_d(X)$ in $Hilb^{dt+1}(X)$ is a compactification which we call the *Hilbert compactification.*

• **Kontsevich compactification**: A stable map is a morphism of a connected nodal curve $f : C \to X$ with finite automorphism group. Here two maps $f : C \to X$ and $f' : C' \to X$ are isomorphic if there exists an isomorphism $\eta : C \to C'$ satisfying $f' \circ \eta = f$. The moduli space $\mathcal{M}_0(X, d)$ of isomorphism classes of stable maps $f : C \to X$ with arithmetic genus 0 and $\deg(f^*\mathcal{O}_X(1)) = d$ has a projective coarse moduli space. The closure $\mathbf{M}_d(X)$ of $R_d(X)$ in $\mathcal{M}_0(X, d)$ is a compactification, called the *Kontsevich compactification.*

• **Simpson compactification**: A coherent sheaf E on X is *pure* if any nonzero subsheaf of E has the same dimensional support as E. A pure sheaf E is called *semistable* if

$$\frac{\chi(E(t))}{r(E)} \leq \frac{\chi(E''(t))}{r(E'')} \qquad \text{for } t >> 0$$

for any nontrivial pure quotient sheaf E'' of the same dimension, where $r(E)$ denotes the leading coefficient of the Hilbert polynomial $\chi(E(t)) = \chi(E \otimes \mathcal{O}_X(t))$. We obtain *stability* if $\leq$ is replaced by $<$. If we replace the quotient sheaves E'' by subsheaves E' and reverse the inequality, we obtain an equivalent definition of (semi)stability. There is a projective moduli scheme $\mathcal{S}imp^P(X)$ of semistable sheaves on X of a given Hilbert polynomial P. If C is a smooth rational curve in X, then the structure sheaf $\mathcal{O}_C$ is a stable sheaf on X. The closure $\mathbf{P}_d(X)$ of $R_d(X)$ in $\mathcal{S}imp^{dt+1}(X)$ is a compactification called the *Simpson compactification.*

2. Rational curves in homogenous varieties

In [**Kie07, CK11, CHK12**], the authors investigated the birational geometry of compactified moduli spaces of rational curves of degree ≤ 3 in homogeneous varieties.

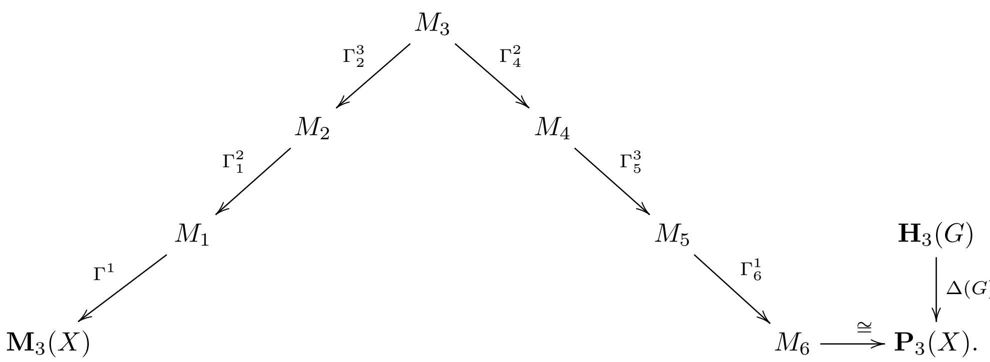
2.1. Rational curves in $\mathbb{P}^3$. The cases $d \leq 2$ has been well-known for experts. The degree 3 case has investigated by using the geometric invariant theoretic quotient ([**Tha96, DH98**]) and the elementary modification of sheaves ([**HL10**]).

THEOREM 2.1. [**CHK12**, Section 4] *Let X be a projective homogenous variety satisfying certain condition in* [**CHK12**, Lemma 2.1].

(1) $\mathbf{H}_3(X)$ *is the smooth blow-up of* $\mathbf{P}_3(X)$ *along the locus* $\Delta(X)$ *of planar stable sheaves.*
(2) $\mathbf{P}_3(X)$ *is obtained from* $\mathbf{M}_3(X)$ *by three weighted blow-ups followed by three weighted blow-downs. In other words,* $\mathbf{P}_3(X)$ *is obtained from* $\mathbf{M}_3(X)$ *by blowing up along* Γ_0^1, Γ_1^2, Γ_2^3 *and then blowing down along* Γ_3^2, Γ_4^3, Γ_5^1 *where* Γ_i^j *is the proper transform of* Γ_{i-1}^j *if* Γ_{i-1}^j *is not the blow-up/-down center and the image/preimage of* Γ_{i-1}^j *otherwise. Here* Γ_0^1 *is the locus of stable maps whose images are lines;* Γ_0^2 *is the locus of stable maps whose images consist of two lines;* Γ_1^3 *is the subvariety of the exceptional divisor* Γ_1^1 *which is a fiber bundle over* Γ_0^1 *with fibers*

$$\mathbb{P}\mathrm{Hom}_1(\mathbb{C}^2, \mathrm{Ext}^1_X(\mathcal{O}_L, \mathcal{O}_L(-1))) \cong \mathbb{P}^1 \times \mathbb{P}\mathrm{Ext}^1_X(\mathcal{O}_L, \mathcal{O}_L(-1))$$

where Hom_1 *denotes the locus of rank 1 homomorphisms.*



2.2. Rational curves in Q_3. Let Q_3 be a smooth quadric hypersurface in $\mathbb{P}^4$. The most natural moduli theoretic view point of the variety Q_3 is the linear section of Grassmannian variety $\mathrm{Gr}(2,4)$ under the under Plücker embedding into $\mathbb{P}(\wedge^2\mathbb{C}^4) = \mathbb{P}^5$. Furthermore, it is known that Q_3 is a homogenous variety and thus we can apply the general result about the homogenous case studied in [**CHK12**]. For example, we can apply the main result of [**CHK12**] for the case $d = 2$.

THEOREM 2.2. *The Hilbert scheme* $\mathbf{H}_2(Q_3)$ *of conics in* Q_3 *is isomorphic to* $\mathrm{Gr}(3,5)$. *Furthermore, there exists a blow-up/down digram between* $\mathbf{H}_2(Q_3)$ *and* $\mathbf{M}_2(Q_3)$. *In special, the Kontsevich compactification* $\mathbf{M}_2(Q_3)$ *is a smooth stack.*

3. Rational curves in V_5

Let V_5 be the intersection of $\mathrm{Gr}(2,5)$ with three hyperplanes in $\mathbb{P}^9$ under Plücker embedding. The Hilbert scheme $\mathbf{H}_d(V_5)$ of rational curves in of degree $d \leq 3$ is isomorphic to the moduli space of stable sheaves $\mathbf{P}_d(V_5)$ by the following Lemma.

LEMMA 3.1. $\mathbf{H}_d(V_5) \cong \mathbf{P}_d(V_5)$ *for* $d \leq 3$.

Proof. The case $d = 1$ or 2 is obvious. Let $F \in \mathbf{P}_3(V_5)$ be a stable sheaf. By Corollary 1.39 in [San14], $F = \mathcal{O}_C$ for some locally CM-curve $C \subset Y$ with Hilbert polynomial $3m + 1$. Hence the natural map

$$\mathbf{H}_3(V_5) \to \mathbf{P}_3(V_5),\ I_C \mapsto \mathcal{O}_C$$

is an isomorphism. □

Thus we have a explicit description of the moduli space $\mathbf{P}_d(V_5)$.

Proposition 3.2 ([Fae05, FN89, Ili94, San14]). $\mathbf{P}_1(V_5) \cong \mathbb{P}^2$, $\mathbf{P}_2(V_5) \cong \mathbb{P}^4$, $\mathbf{P}_3(V_5) \cong \mathrm{Gr}(2, 5)$.

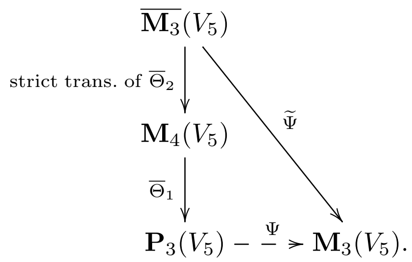
For each d, the universal sheaves of $\mathbf{P}_d(V_5)$ was explicitly presented in [San14, Proposition 2.20, Proposition 2.32, Proposition 2.46]. In the view point of birational geometry, we have

Theorem 3.3. [Chu19] *Let*

$$\Psi : \mathbf{P}_3(V_5) \dashrightarrow \mathbf{M}_3(V_5)$$

be the natural birational map. Then,

(1) the undefined locus of Ψ^{I} *is* $\mathrm{Bs}(\Psi) = \overline{\Theta}_2$.
(2) The map Ψ *extends to a birational regular morphism* $\widetilde{\Psi}$ *by the two times blow-ups of* $\mathbf{P}_3(V_5)$ *along* $\overline{\Theta}_1$ *followed by the strict transform of* $\overline{\Theta}_2$.
(3) The two times blown-up space of $\mathbf{P}_3(V_5)$ *has at most finite group quotient singularity.*



3.1. Moduli space of stable maps in Y. Let $\overline{M}(Y, d)$ be the moduli space of stable maps with degree d and genus 0. By the work in [Chu19, CY19],

Proposition 3.4.

$$\overline{M}(Y, 1) \cong \mathbf{M}_1 = \mathbb{P}^2.$$

For $d = 2, 3$,

$$\overline{M}(Y, d) = \overline{M}(Y, d)_{\mathrm{prin}} \cup \overline{M}(Y, d)_L.$$

Here $\overline{M}(Y, d)_{\mathrm{prin}}$ *is the closure of the locus of stable maps whose image is smooth rational curves of degree d and* $\overline{M}(Y, d)_L$ *is the projective bundle over* $\mathbf{M}_1$ *with fiber* $\overline{M}(\mathbb{P}^1, d)$*. Furthermore, there exists a birational contraction*

$$\Phi_d : \overline{M}(Y, d)_{\mathrm{prin}} \to \mathbf{M}_d$$

where

(1) Φ_2 *is a smooth blow-up along the locus of double lines (which is isomorphic to a conic in* $\mathbf{M}_2 \cong \mathbb{P}^2$*)*
(2) and Φ_3 *is the rational composition of two-times weighted blow-up followed by a small contraction along geometric meaningful centers.*

4. Rational curves in V_{22}

The Fano variety V_{22} can be described by three different ways: Isotropic Grassmannian variety, Moduli space of twisted cubics in $\mathbb{P}^3$ and the Hilbert scheme of points on the dual plane $\mathbb{P}^{2*}$. The first (resp. third) two ones is useful for the line and conic (resp. twisted cubics) in V_{22}.

4.1. The variety of type V_{22} via Isotropic Grassmannian variety.

DEFINITION 4.1. Let $\dim V = 7$ and $\dim N = 3$. Let $\sigma : N \to \wedge^2 V^*$ be a fixed injective (general) alternating form. Let us define by

$$V_{22} = \{[W] \in \mathrm{Gr}(3, V) \mid \sigma(n)(u, v) = 0,\ \forall u, v \in W, \forall n \in N\}.$$

For example, one can choose the alternating forms:

$$\omega_0 := e_0^* \wedge e_4^* + e_1^* \wedge e_3^* - e_2^* \wedge e_6^*$$
$$\omega_1 := e_1^* \wedge e_5^* + e_2^* \wedge e_4^* - e_0^* \wedge e_6^*$$
$$\omega_2 := e_0^* \wedge e_2^* + e_3^* \wedge e_5^* + e_4^* \wedge e_6^*$$

REMARK 4.2. Let $\omega : \mathbb{C} \to \wedge^2 V^*$ be the non-zero alternating form with $\dim V = n$. We define the isotropic Grassmannian by

$$\mathrm{Gr}_\omega(k, n) = \{[W] \in \mathrm{Gr}(k, n) \mid \omega(1)(u, v) = 0, \forall u, v \in W\}.$$

Then, V_{22} is the complete intersection

$$V_{22} = \bigcap_{i=0}^{2} \mathrm{Gr}_{\sigma(n_i)}(3, 7)$$

of isotropic Grassmannians $\mathrm{Gr}_{\sigma(n_i)}(3, 7)$ with the general two forms $\sigma(n_i)$ on V.

4.2. Rational curves in the Grassmann $\mathrm{Gr}(3, 7)$.

Let us denote by $S(C_1, C_2, \cdots, C_n)$ the *rational normal scroll* arising from the rational normal curves C_i (allowing C_i to be a point) with fixed isomorphisms $\mathbb{P}^1 \cong C_j$, $1 \leq j \leq n$ (whenever C_i is not a point). Grothendieck's theorem says that any vector bundle on the projective line $\mathbb{P}^1$ splits into the direct sum of line bundles. If $f : \mathbb{P}^1 \to \mathrm{Gr}(3, 7)$ is a morphism of degree d, the pullback of the (dual) universal bundle gives a surjective homomorphism

$$\varphi : \mathcal{O}_{\mathbb{P}^1}^{\oplus 7} \to \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2) \oplus \mathcal{O}_{\mathbb{P}^1}(d_3)$$

over $\mathbb{P}^1$ with $d = d_1 + d_2 + d_3, d_1 \leq d_2 \leq d_3$.

- When $d = 1$, we have $d_1 = d_2 = 0$ and $d_3 = 1$. The composition $\pi \circ \varphi : \mathcal{O}^{\oplus 7} \to \mathcal{O}$ of φ with the projection onto the first (second) factor is surjective and gives us the point $p_1, p_2 \in \mathbb{P}^6$ while that with the second projection $\pi \circ \varphi$ gives a line in $\mathbb{P}^6$. Thus, a line in $\mathrm{Gr}(3, 7)$ parameterizes the 2-dimensional planes H in $\mathbb{P}^6$ passing through p_1, p_2 as varying the points $p \in l$.
- When $d = 2$, we have $(d_1, d_2, d_3) = (0, 1, 1)$ or $(0, 0, 2)$. In the first case, $\pi_1 \circ \varphi$ gives a point p while $\pi_2 \circ \varphi$ and $\pi_3 \circ \varphi$ give lines l, l'. In the second case, both $\pi_1 \circ \varphi$ and $\pi_2 \circ \varphi$ give us points while the third one gives us a conic C in $\mathbb{P}^6$. Hence, the general conics in $\mathrm{Gr}(3, 7)$ parameterizes planes passing through three points $p, q, \phi(q)$ such that $q \in l$, $\phi(q) \in l'$ for the induced isomorphism $\phi : l \cong l'$. That is, the conic parameterizes the variety of 2-dimensional planes of ruling of the scroll $S(p, l, l')$.
- When $d = 3$, we have $(d_1, d_2, d_3) = (1, 1, 1)$ or $(0, 1, 2)$ or $(0, 0, 3)$. In the first case, the projections give us three lines l, l', l'' in $\mathbb{P}^6$ of general position. That is, the twisted cubics parameterizes the variety of 2-dimensional planes of ruling of the scroll $S(l, l', l'')$.

For the case $d = 2$, the general degree 2 map $f : \mathbb{P}^1 \to \mathrm{Gr}(3, 7)$ associates to the point p which is the intersection of planes presenting the value $f(x)$ for all $x \in \mathbb{P}^1$ and $\mathbb{P}^4$ which is the union of planes presenting the value $f(x)$ for all $x \in \mathbb{P}^1$. Hence we have a map into the partial flag variety $\mathrm{Fl}(1, 5, 7)$ as follows.

$$R_2(\mathrm{Gr}(3, 7)) \dashrightarrow \mathrm{Fl}(1, 5, 7),\ [f] \mapsto [(p, \mathbb{P}^4)], p \in \mathbb{P}^4.$$

This observation is the key point of description of $\mathbf{M}_2(V_{22})$ ([She10, Appendix A]).

For the case $d = 3$, the variety $S(l, l', l'')$ is called by a Segre 3-fold where the linear spanning of $S(l, l', l'')$ is a $\mathbb{P}^5 = \langle S(l, l', l'') \rangle$ in $\mathbb{P}^6$. Note that each Segre 3-fold can be obtained by orbits of the automoprhism group PGL(6) of $\mathbb{P}^5$. On the other hand, the Segre 3-fold in $\mathbb{P}^5$ can be defined as the image of the embedding

$$\Sigma := \mathbb{P}^1 \times \mathbb{P}^2 \stackrel{|\mathcal{O}(1,1)|}{\longrightarrow} \mathbb{P}^2$$

given by the complete linear system $|\mathcal{O}(1,1)|$. The defining equations of the image $\sqrt{-1}(\Sigma)$ is given by

$$\text{rank} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{pmatrix} \leq 1$$

up to the projective motion. Here $[x_0 : x_1 : x_2 : \cdots : x_5]$ is the homogeneous coordinate of $\mathbb{P}^5$.

4.3. Lines and conics on V_{22}.

PROPOSITION 4.3. [**KPS18**, Proposition 5.4.4] *The Hilbert scheme $\mathbf{P}_1(V_{22})$ is isomorphic to a singular quartic plane curve.*

Let $\mathcal{U}$ and $\mathcal{Q}$ be the pull-back of the universal bundles on $\mathrm{Gr}(3, V)$ such that

$$0 \to \mathcal{U} \to V \otimes \mathcal{O}_X \to \mathcal{Q} \to 0.$$

Note that $\mathrm{Hom}(\mathcal{U}, \mathcal{Q}^*) \cong N$.

PROPOSITION 4.4. *The Hilbert scheme of conics on V_{22} is isomorphic to $\mathbf{H}_2 \cong \mathbb{P}(N^*) = \mathbb{P}^2$.*

The original description of conics was studied by Mukai (as cited by M. Shen). For the the detail, see [**She10**, Appendix A].

4.4. Cubic curves on V_{22}. Twisted cubic curves in V_{22} has a simple descrption via the Hilbert scheme of twisted cubics in $\mathbb{P}^3$.

PROPOSITION 4.5. [**CL20**] *Let $\mathbf{M}_3$ be the moduli space of stable sheaves with the Hilbert polynomial $3m+1$. Then,*

(a) $\mathbf{M}_3$ is smooth.
*(b) $\mathbf{M}_3$ is isomorphic to the projective space $\mathbb{P}(V^*)$ ([**KS04**]).*
(c) The locus of stable sheaves supported on lines in V_{22} is an unique point in $\mathbb{P}(V^*)$.

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SOME INVARIANTS OF PROJECTIVE VARIETIES

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Abstract. A nondegenerate projective variety $X \subset \mathbb{P}^r_{\mathbb{K}}$ is defined by the homogeneous ideal I_X in the homogeneous coordinate ring $\mathbb{K}[x_0, x_1, \dots, x_r]$ of $\mathbb{P}^r_{\mathbb{K}}$. That is, X is the set of common roots of generators of I_X. Classically, it is one of the most fundamental problem in projective algebraic geometry to understand the relation between geometric properties of X and algebraic properties of I_X. In this short note, I will introduce some projective invariants of projective varieties related to my research topic.

1. Minimal Free Resolution

For the graded ring $S = \mathrm{K}[X_0, X_1, \dots, X_r] = \oplus_{d\in\mathbb{Z}} S_d$, let $M = \oplus_{d\in\mathbb{Z}} M_d$ be a given nonzero graded S-module. To explain the number of generators of M and relations between the generators, it is natural to study the minimal free resolution of M. In this section, we recall "*Hilbert Syzygy Theorem*" and some cohomological tools related to the Koszul complex. Consider the minimal graded free resolution of M:

$$\cdots \longrightarrow \oplus_j S^{\beta_{i,j}}(-i-j) \xrightarrow{\varphi_i} \cdots \longrightarrow \oplus_j S^{\beta_{1,j}}(-1-j) \xrightarrow{\varphi_1} \oplus_j S^{\beta_{0,j}}(-j) \xrightarrow{\varphi_0} M \longrightarrow 0$$

Precisely we prove that the length of the graded minimal free resolution of M is less than or equal to $r+1$(Hilbert Syzygy Theorem) and also we explain how to compute the Betti number $\beta_{i,j}$ (M. Green's sequence, see Theorem **??**).

Lemma 1.1. *Let $m = (X_0, X_1, \dots, X_r)$. Then for $S/m \cong K$, we have*

$$\beta_{i,j} = dim_K Tor_i(M, K)_{i+j}$$

where $Tor_i(M, K)_{i+j}$ is the $(i+j)$-th piece of the graded S-module

$$Tor_i(M, K) = Tor_i(M, K)_0 \oplus Tor_i(M, K)_1 \oplus Tor_i(M, K)_2 \cdots .$$

Proof. By tensoring the above resolution with $S/m \cong \mathrm{K}$, we have the following complex

$$\cdots \longrightarrow \oplus_{j\in\mathbb{Z}} S^{\beta_{i,j}}(-i-j) \otimes K \longrightarrow \cdots \longrightarrow \oplus_{j\in\mathbb{Z}} S^{\beta_{1,j}}(-1-j) \otimes \mathrm{K}$$

$$\xrightarrow{\widetilde{\varphi_1}} \oplus_{j\in\mathbb{Z}} S^{\beta_{0,j}}(-j) \otimes \mathrm{K} \xrightarrow{\widetilde{\varphi_0}} M \otimes \mathrm{K} \longrightarrow 0$$

First, note that the boundary map

$$\widetilde{\varphi_0} : \oplus_{j\in\mathbb{Z}} S^{\beta_{0,j}}(-j) \otimes \mathrm{K} \longrightarrow M \otimes \mathrm{K}$$

is the zero map by minimality. For $\ell \geq 1$, define M_ℓ to be the kernel of $\varphi_{\ell-1}$ and consider the following diagram:

$$\begin{array}{ccccc} \cdots \longrightarrow \oplus_{j\in\mathbb{Z}} S^{\beta_{\ell,j}}(-\ell-j) & & \xrightarrow{\varphi_\ell} & & \oplus_{j\in\mathbb{Z}} S^{\beta_{\ell-1,j}}(-\ell+1-j) \xrightarrow{\varphi_{\ell-1}} \cdots \\ & \searrow & & \nearrow & \\ & & M_\ell & & \\ & \nearrow & & \searrow & \\ 0 & & & & 0 \end{array}$$

In the commutative diagram

$$\begin{array}{ccccc} \oplus_{j\in\mathbb{Z}} S^{\beta_{\ell,j}}(-\ell-j) \otimes \mathrm{K} & & \xrightarrow{\widetilde{\varphi_\ell}} & & \oplus_{j\in\mathbb{Z}} S^{\beta_{\ell-1,j}}(-\ell+1-j) \otimes \mathrm{K} \\ & \phi_\ell \searrow & & \nearrow & \\ & & M_\ell \otimes \mathrm{K} & & \end{array}$$

ϕ_ℓ is the zero map by minimality and thus it is shown that $\widetilde{\varphi_\ell}$ is the zero map for all $\ell \geq 0$. Therefore

$$Tor_\ell(M, \mathrm{K}) \cong \oplus_{j\in Z} S^{\beta_{\ell,j}}(-\ell-j) \otimes \mathrm{K}.$$

Since $\mathrm{K} \cong S/m$, the $(i+j)$-th piece of $Tor_i(M, \mathrm{K})$ is $S^{\beta_{i,j}}(-i-j) \otimes \mathrm{K} = \mathrm{K}^{\beta_{i,j}}$. □

Lemma 1.1 says that the graded Betti number of M can be interpreted by the dimension of graded pieces of $Tor_i(M, \mathrm{K})$. Recall that since $Tor_i(-,-)$ is a covariant functor, i.e., $Tor_i(M, \mathrm{K}) \cong Tor_i(\mathrm{K}, M)$, we can compute $Tor_i(M, \mathrm{K})_{i+j}$ by using the well-known Koszul type minimal free resolution of $\mathrm{K} \cong S/m$:

$$0 \longrightarrow \wedge^{r+1} V \otimes S(-r-1) \longrightarrow \cdots \longrightarrow \wedge^2 V \otimes S(-2) \longrightarrow V \otimes S(-1) \longrightarrow S \longrightarrow S/m \longrightarrow 0$$

where V is the $(r+1)$-dimensional K-vector space generated by $X_0, X_1, \ldots, X_r$. This complex is just the "Koszul complex" with respect to the regular system $X_0, X_1, \ldots, X_r \in m$. Above Koszul complex enables us to prove the following:

THEOREM 1.2 (Hilbert Syzygy Theorem). *Let M be a finitely generated graded $S = K[X_0, X_1, \ldots, X_r]$-module. Then the length of the minimal free resolution of M is $\leq r+1$.*

PROOF. By Lemma 1.1, it is enough to show that for all $i \geq r+2$,

$$\beta_{i,j} = dim_{\mathrm{K}} Tor_i(M, \mathrm{K})_{i+j} = 0.$$

Since the length of the minimal free resolution of K is $\leq r+1$ and since $Tor_i(-,-)$ is a covariant functor,

$$Tor_i(M, \mathrm{K}) = Tor_i(\mathrm{K}, M) = 0 \text{ for all } i \geq r+2,$$

so we have the desired result. □

Now we turn to the geometric cases. For a nonzero coherent sheaf $\mathcal{F}$ on $\mathcal{P}^r$, consider the associated graded S- module

$$F = \oplus_{\ell\in\mathbb{Z}} H^0(\mathcal{P}^r, \mathcal{F}(\ell))$$

and the minimal free resolution

$$\cdots \longrightarrow \oplus_{j\in\mathbb{Z}} S^{\beta_{i,j}}(-i-j) \longrightarrow \cdots \longrightarrow \oplus_{j\in\mathbb{Z}} S^{\beta_{1,j}}(-1-j) \longrightarrow \oplus_{j\in\mathbb{Z}} S^{\beta_{0,j}}(-j) \longrightarrow F \longrightarrow 0.$$

By using the Euler sequence

$$0 \longrightarrow \mathcal{M} := \Omega_{\mathcal{P}^r}(1) \longrightarrow V \otimes \mathcal{O}_{\mathcal{P}^r} \longrightarrow \mathcal{O}_{\mathcal{P}^r}(1) \longrightarrow 0$$

where $V = H^0(\mathcal{P}^r, \mathcal{O}_{\mathcal{P}^r}(1))$, M. Green[G] obtained the following general connection between syzygies and some cohomology groups:

2. Varieties of minimal degree

Varieties with $deg(X) = codim(X) + 1$ are called *varieties of minimal degree*. More than 100 years ago, they were classified by P.Del Pezzo and E.Bertini as following

- $\mathcal{P}^r$
- quadric hypersurface
- (a cone over) the veronese surface in $\mathcal{P}^5$
- (a cone over) rational normal scroll.

A modern proof of this classification can be found in [**EH**] and [**F**]. Varieties of minimal degree are always 2-regular, thus homogeneous ideal I_X of X is generated by quadric equations and satisfies Property N_p for all $p \geq 0$. Also these varieties are arithmetically Cohen-Macaulay and their Betti diagrams are well-known as follows;

Lemma 2.1. *Let $X \subset \mathcal{P}^r_K = Proj(S)$ be a variety of minimal degree. Setting $c = codim(X)$, X has minimal free resolution of the form*

$$0 \to S(-c-1)^{\beta_{c,1}} \to S(-c)^{\beta_{c-1,1}} \to \cdots \to S(-3)^{\beta_{2,1}} \to S(-2)^{\beta_{1,1}} \to I_X \to 0$$

where $\beta_{i,1} = i \cdot \binom{c+1}{i+1}$.

Proof. Since X is arithmetically Cohen-Macaulay, using hyperplane section method in the section 2.4 it suffices to consider rational normal curve. Let $\nu_d : \mathcal{P}^1 \longrightarrow \mathcal{P}^d$ be a rational normal curve of degree d. For given Euler sequence

$$0 \longrightarrow \mathcal{M}_d \longrightarrow V \otimes \mathcal{O}_{\mathcal{P}^1} \longrightarrow \mathcal{O}_{\mathcal{P}^1}(d) \longrightarrow 0,$$

we have the following exact sequence of cohomology groups

$$\wedge^{i+2} V \otimes H^1(\mathcal{P}^1, \mathcal{O}_{\mathcal{P}^1}(d(j-2))) \longrightarrow H^1(\mathcal{P}^1, \wedge^{i+1}\mathcal{M}_d \otimes \mathcal{O}_{\mathcal{P}^1}(d(j-1))) \longrightarrow H^2(\mathcal{P}^1, \wedge^{i+2}\mathcal{M}_d \otimes \mathcal{O}_{\mathcal{P}^1}(d(j-2))).$$

Since by serre duality, $H^1(\mathcal{P}^1, \mathcal{O}_{\mathcal{P}^1}(d(j-2))) = 0$ for $j \geq 2$ and since by dimension counting, $H^2(\mathcal{P}^1, \wedge^{i+2}\mathcal{M}_d \otimes \mathcal{O}_{\mathcal{P}^1}(d(j-2))) = 0$. Therefore the Betti number $\beta_{i,j} = h^1(\mathcal{P}^1, \wedge^{i+1}\mathcal{M}_d \otimes \mathcal{O}_{\mathcal{P}^1}(d(j-1)))$ be zero for all i and for all $j \geq 2$.

Now we shall compute the $\beta_{i,1}$. Since $\nu_d(\mathcal{P}^1)$ is projective normal, $\beta_{o,1} = 0$ and $\beta_{i,1} = h^1(\mathcal{P}^1, \wedge^{i+1}\mathcal{M}_d)$ for $i \geq 1$. Note that locally free sheaf of rank d on $\mathcal{P}^1$ is isomorphic to a direct sum of invertible sheaves, see Hartshorn's textbook[**Hart**], Exercise V.2.6. So we represent $\mathcal{M}_d$ to the form,

$$\mathcal{M}_d = \underbrace{\mathcal{O}_{\mathcal{P}^1}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathcal{P}^1}(-1)}_{d-times} = (\mathcal{O}_{\mathcal{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathcal{P}^1}) \otimes \mathcal{O}_{\mathcal{P}^1}(-1).$$

Let $W = \mathcal{O}_{\mathcal{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathcal{P}^1}$ as the vector space of dimension d, then $\wedge^{i+1}\mathcal{M}_d = \wedge^{i+1} W \otimes \mathcal{O}_{\mathcal{P}^1}(-i-1)$. Since by Serre duality, $h^1(\mathcal{P}^1, \mathcal{O}_{\mathcal{P}^1}(-i-1)) = h^0(\mathcal{P}^1, \mathcal{O}_{\mathcal{P}^1}(i-1))$. Therefore

$$\begin{aligned} \beta_{i,1} = & \quad h^1(\mathcal{P}^1, \wedge^{i+1}\mathcal{M}_d) & = h^1(\mathcal{P}^1, \wedge^{i+1} W \otimes \mathcal{O}_{\mathcal{P}^1}(-i-1)) \\ = & dim_K\{\wedge^{i+1} W\} \cdot h^1(\mathcal{P}^1, \mathcal{O}_{\mathcal{P}^1}(-i-1)) = & \binom{d}{i+1} \cdot h^0(\mathcal{P}^1, \mathcal{O}_{\mathcal{P}^1}(i-1)) \\ = & \quad \binom{d}{i+1} \cdot i & . \end{aligned}$$

□

2.1. Rational normal scroll.

Definition 2.2. For $1 \leq a_1 \leq \cdots \leq a_n$, if $\mathcal{E} = \mathcal{O}_{\mathcal{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathcal{P}^1}(a_n)$, $X = \mathcal{P}(\mathcal{E})$, and $L = \mathcal{O}_{\mathcal{P}(\mathcal{E})}(1)$, $H^0(X, L) = H^0(\mathcal{P}^1, \mathcal{E}) = a_1 + \cdots + a_n + n$ then
$\varphi_{|L|} : X \longrightarrow \mathcal{P}^{a_1+\cdots+a_n+n-1}$ is embedding. $S(a_1, \ldots, a_n) := \varphi_{|L|}(X) \subset \mathcal{P}^{a_1+\cdots+a_n+n-1}$ is rational normal scroll.

(i) $X = \bigcup_{x \in \mathcal{P}^1} \Pi^{-1}(x)$.
(ii) $\Pi^{-1}(x) \cong \mathcal{P}^{n-1}$.
(iii) $L\, |_{\Pi^{-1}(x)} \cong \mathcal{O}_{\mathcal{P}^{n-1}}(1)$.
Where $\Pi : X \longrightarrow \mathcal{P}^1$ is natural projection.

Definition 2.3. Let $r + 1 = a_1 + \cdots + a_n + n$ and $\Lambda_1 \cong \mathcal{P}^{a_1}, \ldots, \Lambda_n \cong \mathcal{P}^{a_n}$ are skew subspaces of $\mathcal{P}^r$. $C_i \subset \Lambda_i$ for $1 \leq i \leq n$ are rational normal curve of degree a_i defined by the isomorphism $\varphi_i : \mathcal{P}^1 \longrightarrow \Lambda_i$. Then the rational normal scroll $S(a_1, \ldots, a_n) := \bigcup_{x \in \mathcal{P}^1} < \varphi_1(x), \ldots, \varphi_n(x) >$.

Definition 2.4. Let $x_0, \ldots, x_r$ be homogeneous coordinate on $\mathcal{P}^r$. Consider the following $2 \times r - n + 1$ matrix M,

$$\left(\begin{array}{cccc|cccc|ccc|cccc} x_0 & \cdot & \cdot & x_{a_1-1} & x_{a_1+1} & \cdot & \cdot & x_{a_1+a_2} & \cdot & \cdot & \cdot & x_{r-a_n} & \cdot & \cdot & x_{r-1} \\ x_1 & \cdot & \cdot & x_{a_1} & x_{a_1+2} & \cdot & \cdot & x_{a_1+a_2+1} & \cdot & \cdot & \cdot & x_{r-a_n+1} & \cdot & \cdot & x_r \end{array} \right).$$

Then the rational normal scroll $S(a_1, \ldots, a_n) \subset \mathcal{P}^r$ is defiend as the rank 1 locus of M.

Theorem 2.5. *Above three definitions are equivalent.*

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INTRODUCTION TO FLAG DOMAINS

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Abstract. In this note, we introduce flag domains and some examples.

Let Δ be the unit disc in the complex plane $\mathbb{C}$,

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

and $\mathbb{P}^1$ be the Riemann surface. Then $\mathbb{C} = \mathbb{P}^1 - \{\infty\}$ is embedded in $\mathbb{P}^1$ by the stereographic projection, and Δ is the open lower hemisphere of $\mathbb{P}^1$. Here $\Delta \subset \mathbb{P}^1$, $\mathbb{P}^1$ are the simplest examples of flag domain and flag manifold respectively.

In the general context flag domains are defined by the following: let $G^{\mathbb{C}}$ be a complex semisimple Lie group. Let P be a parabolic subgroup in $G^{\mathbb{C}}$ and G a non-compact real form of $G^{\mathbb{C}}$. Then an open G-orbit in the flag manifold $G^{\mathbb{C}}/P$ is called a *flag domain* (cf. [3, 4]). In 1960's to 1970's, Wolf studied G-orbits on $G^{\mathbb{C}}/P$ using representation theory and Lie algebraic tools intensively. One of the consequence of his work is the following:

Theorem 0.1 (Wolf, [4]). *There are only finitely many G-orbits on X. The maximal dimensional orbits are open and the minimal-dimensional orbits are closed.*

As a result, open G-orbits on X always exists. Without loss of generality, assume that the G-orbit of eP is open in $G^{\mathbb{C}}/P$. Then we call a flag domain $F = G \cdot z_0 = G/V$ with $z_0 = eP \in G^{\mathbb{C}}/P$ a *canonical flag domain* if $V := G \cap P$ is a compact Lie subgroup containing a maximal torus of G. For the unit disc and the Riemann sphere $G^{\mathbb{C}}$, P, G are given by

$$G^{\mathbb{C}} = SL(2, \mathbb{C}) = \{\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1\}$$

which is the complex semisimple Lie group of dimension 3,

$$P = \{\pm \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : ad = 1\},$$

$$G = SU(1,1) = \{\pm \begin{pmatrix} a & b \\ \pm\bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1\},$$

$$V = \{\pm \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : |a| = 1\}.$$

$G^{\mathbb{C}}$ acts on $\mathbb{P}^1$ by the linear transformation and G acts on Δ by the linear fractional transformations

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

0.2. Measurable flag domains. A flag domain $X_0 \subset X$ is called **measurable** if it carries a G-invariant volume element. Most of the flag domains which are investigated deeply linked to complex geometry, algebraic geometry or several complex variables are measurable.

THEOREM 0.2 (Wolf, [4]). *Let $G \cdot z_0 \subset X$ be an open orbit in the complex flag manifold $X = G/P$. Then the following conditions are equivalent:*

(1) The orbit $G \cdot z_0$ is measurable.
(2) $G \cap P$ is the G-centralizer of a compact torus subgroup of G.
(3) $G \cdot z_0$ has a G-invariant, possibly indefinite, Kähler metric, thus a G-invariant measure obtained from the volume form of this metric.

0.3. Generalized type I domains. For any two positive integers p and q, let $H_{p,q}$ be the standard non-degenerate Hermitian form of signature (p, q) on $\mathbb{C}^{p+q}$ where p eigenvalues are 1 and q eigenvalues are -1, represented by the matrix $\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ under the standard coordinates. For a positive integer $r < p+q$, denote by $Gr(r, \mathbb{C}^{p+q})$ the Grassmannian of r-dimensional complex linear subspaces (or simply *r-planes*) of $\mathbb{C}^{p+q}$. When $1 \le r \le p$, we define the domain $D^r_{p,q}$ in $Gr(r, \mathbb{C}^{p+q})$ to be the set of positive definite r-planes in $\mathbb{C}^{p+q}$ with respect to $H_{p,q}$. We call $D^r_{p,q}$ a *generalized type-I domain*. The generalized type-I domain $D^r_{p,q}$ is an $SU(p, q)$-orbit on $Gr(r, \mathbb{C}^{p+q})$ under the natural action induced by that of $SL(p + q; \mathbb{C})$ on $Gr(r, \mathbb{C}^{p+q})$. Recently $D^r_{p,q}$ have been studied regarding the proper holomorphic mappings between them and regarding the CR maps on some CR manifolds in $\partial D^r_{p,q}$. We remark that the case $r = p$ corresponds to the classical bounded symmetric domains of type-I which will be explained in the next section. On the other extreme, when $r = 1$ corresponds to the domains $D_{p,q} := D^1_{p,q}$, which are called the generalized balls. It follows immediately from our definition that $D_{p,q}$ can be also defined as the following domain on $\mathbb{P}^{p+q-1}$:

$$D_{p,q} = \left\{[z_1, \ldots, z_{p+q}] \in \mathbb{P}^{p+q-1} : |z_1|^2 + \cdots + |z_p|^2 > |z_{p+1}|^2 + \cdots + |z_{p+q}|^2\right\}.$$

When $p = 1$, it is biholomorphic to the unit ball in the Euclidean space $\mathbb{C}^q$. The generalized ball is one of the simplest kinds of domains on the projective space and their boundaries are smooth Levi non-degenerate (but not pseudoconvex in general) CR manifolds.

0.4. Bounded symmetric domains. A bounded domain Ω in the complex Euclidean space is called *symmetric* if for each $p \in \Omega$, there exists a holomorphic automorphism I_p such that

(1) I_p^2 is the identity map of Ω,
(2) I_p has p as an isolated fixed point.

All bounded symmetric domains are homogeneous domains, i.e. the automorphism group acts transitively on the domain. In 1920's, E. Cartan classified all irreducible bounded symmetric domains which consist of 4 classical types and 2 exceptional types ([2]).

(1) Type I : $\Omega^I_{m,n} = \{Z \in M(m, n, \mathbb{C}) : I_n - ZZ^* > 0\}$ where $m \ge n = \mathrm{rank}(\Omega^I_{m,n})$
(2) Type II : $\Omega^{II}_m = \{Z \in M(m, m, \mathbb{C}) : I_m - ZZ^* > 0,\ Z^t = -Z\}$, $\mathrm{rank}(\Omega^{II}_m) = [\frac{1}{2}m]$
(3) Type III : $\Omega^{III}_m = \{Z \in M(m, m, \mathbb{C}) : I_m - ZZ^* > 0,\ Z^t = Z\}$, $\mathrm{rank}(\Omega^{III}_m) = m$
(4) Type IV : $\Omega^{IV} = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : ||z||^2 < 2,\ ||z||^2 < 1 + |\frac{1}{2}\sum z_k^2|^2\}$, $\mathrm{rank}(\Omega^{IV}) = 2$,
(5) Type V : $\Omega^V_{16} = \{z \in M^{\mathbb{O}_\mathbb{C}}_{1,2} : 1 - (z|z) + (z^\#|z^\#) > 0,\ 2 - (z|z) > 0\}$, and
(6) Type VI : $\Omega^{VI}_{27} = \{z \in H_3(\mathbb{O}_\mathbb{C}) : 1 - (z|z) + (z^\#|z^\#) - |\det z|^2 > 0,\ 3 - 2(z|z) + (z^\#|z^\#) > 0,\ 3 - (z|z) > 0\}$.

Note that $\Omega_{m,1}$ is the m-dimensional unit ball and irreducible bounded symmetric domain of rank 1 is the unit ball.

Actually bounded symmetric domains are Harish-Chandra realization into the complex Euclidean space. Let X be an irreducible Hermitian symmetric space of non-compact type. Let G be the identity component of the isometry group of X with respect to the Bergman metric of X and $K \subset G$ the isotropy subgroup at $o \in X$. Then X is biholomorphic to G/K. Denote by $\mathfrak{g}$ and by $\mathfrak{k}$ the Lie algebras of G and K respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the Cartan decomposition. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C}$, and $G^{\mathbb{C}}$ be the complex Lie group corresponding to $\mathfrak{g}^{\mathbb{C}}$. Let $\mathfrak{g}_c = \mathfrak{k} + \sqrt{-1}\mathfrak{m}$ be a Lie algebra of compact type and G_c the corresponding connected Lie group of $\mathfrak{g}_c$. Then $\widehat{X} = G_c/K$ is the compact dual of X. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. Note that $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let Δ denote the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$ and let $\mathfrak{g}^{\alpha}$ denote the root space with respect to a root $\alpha \in \Delta$. Let $\Delta_{\mathfrak{k}}$, $\Delta_{\mathfrak{m}}$ denote the set of compact, non-compact roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to the Cartan decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{\mathbb{C}}$ respectively and choose an order of Δ such that the set of positive non-compact roots $\Delta_{\mathfrak{m}}^{+}$ satisfies that $\mathfrak{m}^{+} := \sum_{\alpha \in \Delta_{\mathfrak{m}}^{+}} \mathfrak{g}^{\alpha} = T_o^{1,0}X$. Here $T^{1,0}X$ denotes the holomorphic tangent bundle of X. Denote $\mathfrak{m}^{-} := \sum_{\alpha \in \Delta_{\mathfrak{m}}^{-}} \mathfrak{g}^{\alpha}$. Let M^{+} and M^{-} be the corresponding analytic subgroups in $G^{\mathbb{C}}$. Note that $\mathfrak{m}^{+}$ and $\mathfrak{m}^{-}$ are abelian subalgebras of $\mathfrak{g}^{\mathbb{C}}$. The center $\mathfrak{z}$ of $\mathfrak{k}$ contains an element Z such that $\mathrm{Ad}_Z E = \pm iE$ for $E \in \mathfrak{m}^{\mp}$. $J := \mathrm{Ad}_Z$ is a complex structure on $\mathfrak{m}$. A basis of $\mathfrak{m}$ is given by the elements $X_\alpha = E_\alpha + E_{-\alpha}$ and $Y_\alpha = -i(E_\alpha - E_{-\alpha})$ where α is non-compact positive. For such α, we have the relations $JX_\alpha = Y_\alpha$, $JY_\alpha = -X_\alpha$ and $[X_\alpha, Y_\alpha] = 2iH_\alpha$. Define $X_\alpha^c = iX_\alpha$ and $Y_\alpha^c = iY_\alpha$. Those define a basis of $i\mathfrak{m}$. $K^{\mathbb{C}}$ denoting the analytic subgroup corresponding to $\mathfrak{k}^{\mathbb{C}}$, $K^{\mathbb{C}} \cdot M^{+}$ is a semidirect product. $\widehat{X} = G_c/K$ is identified with $G^{\mathbb{C}}/K^{\mathbb{C}} \cdot M^{+}$ by the identity map of G into $G^{\mathbb{C}}$. For $o = eK \in \widehat{X}$, the orbit $G \cdot o$ is the image of the holomorphic embedding $gK \mapsto g(o)$ of X into $\widehat{X}$ (Borel embedding). The map $\xi \colon \mathfrak{m}^{-} \to \widehat{X}$ defined by

$$\xi(E) = \exp(E)(o)$$

is a holomorphic homeomorphism onto a dense open subset and ξ is Ad_K-equivariant. Then $\Omega = \xi^{-1}(G(o))$ is a bounded symmetric domain in $\mathfrak{m}^{-}$; this is the Harish-Chandra realization of X.

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A JOURNEY FROM DISTRIBUTIVITY TO SET-THEORETIC YANG-BAXTER HOMOLOGY

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Abstract. Distributive structures have been studied for a long time, and especially the importance of seif-distributivity has already been emphasized. Yang-Baxter equation, one of the basic equations in mathematical physics, can be regarded as a generalization of self-distributivity. We introduce distributive structures related to the set-theoretic Yang-Baxter equation, their homology theories, and applications to knot theory.

1. Introduction

The Yang-Baxter equation has played an important role in various fields such as quantum group theory, braided categories, and low-dimensional topology since it was first introduced independently in a study of theoretical physics by Yang [16] and statistical mechanics by Baxter [1]. In particular, since the discovery of the Jones polynomial [8] in 1984, it has been extensively studied in knot theory[1]. As a homological approach, Carter, Elhamdadi, and Saito [2] defined a (co)homology theory for set-theoretic Yang-Baxter operators, from which they provided a method to generate link invariants, and further developments were made by Przytycki [13]. Meanwhile, Joyce [9] and Matveev [10] independently introduced a self-distributive algebraic structure[2], called a quandle, which satisfies axioms motivated by the Reidemeister moves, and it has been generalized as a biquandle. Quandles and biquandles are solutions of the set-theoretic Yang-Baxter equation, which have been used to define homotopical and homological invariants of knots and links [11, 17, 3].

2. Yang-Baxter equation

Let k be a commutative ring with unity and X be a set. We denote by V the free k-module generated by X. Then, a k-linear map $R : V \otimes V \to V \otimes V$ is called a *pre-Yang-Baxter operator* if it satisfies the equation of the following maps $V \otimes V \otimes V \to V \otimes V \otimes V$:

$$(R \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes R) \circ (R \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes R) \circ (R \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes R).$$

We call a pre-Yang-Baxter operator R a *Yang-Baxter operator* if it is invertible.

The classification of the solutions of the Yang-Baxter equation has been actively studied. Following the study by Drinfel′d [4], the set-theoretic solutions of the Yang-Baxter equation have been the focus of various studies [5, 6].

[1]It is known that a certain solution of the Yang-Baxter equation gives rise to the Jones polynomial [8, 14].
[2]The importance of (right) self-distributivity stuructures was already emphasized by Peirce [12].

Definition 2.1. For a given set X, a function $R : X \times X \to X \times X$ satisfying the following equation (called a *set-theoretic Yang-Baxter equation*)

$$(R \times \mathrm{Id}_X) \circ (\mathrm{Id}_X \times R) \circ (R \times \mathrm{Id}_X) = (\mathrm{Id}_X \times R) \circ (R \times \mathrm{Id}_X) \circ (\mathrm{Id}_X \times R)$$

is called a *set-theoretic pre-Yang-Baxter operator* or a *set-theoretic solution of the Yang-Baxter equation*. In addition, if R is invertible, then we call R a *set-theoretic Yang-Baxter operator*.

Special families known as *biracks* and *biquandles* are strongly related to the knot theory. Their precise definitions are as follows:

Definition 2.2. For a given set X, let R be a set-theoretic Yang-Baxter operator denoted by

$$R(A_1, A_2) = (R_1(A_1, A_2), R_2(A_1, A_2)) = (A_3, A_4),$$

where $A_i \in X$ $(i = 1, 2, 3, 4)$, and $R_j : X \times X \to X$ $(j = 1, 2)$ are binary operations. We consider the following conditions:

(1) For any $A_1, A_3 \in X$, there exists a unique $A_2 \in X$ such that $R_1(A_1, A_2) = A_3$. In this case, R_1 is *left-invertible*.
(2) For any $A_2, A_4 \in X$, there exists a unique $A_1 \in X$ such that $R_2(A_1, A_2) = A_4$. In this case, R_2 is *right-invertible*.
(3) For any $A_1 \in X$, there is a unique $A_2 \in X$ such that $R(A_1, A_2) = (A_1, A_2)$.

The algebraic structure (X, R_1, R_2) is called a *birack* if it satisfies the conditions (1) and (2). A birack is a *biquandle* if the condition (3) is also satisfied.

Remark 2.3. The condition (3) in Definition 2.2 implies that for any $A_2 \in X$, there is a unique $A_1 \in X$ such that $R(A_1, A_2) = (A_1, A_2)$. See Remark 3.3 in [2].

Example 2.4. (1) Let C_n be the cyclic rack of order n, i.e., the cyclic group $\mathbb{Z}_n$ of order n with the operation $i * j = i + 1 \pmod{n}$. Then the function $R : X \times X \to X \times X$ defined by

$$R(i, j) = (R_1(i, j), R_2(i, j)) = (j * i, i \bar{*} j) = (j + 1, i - 1)$$

forms a set-theoretic Yang-Baxter operator. Moreover, (C_n, R_1, R_2) is a biquandle, called a *cyclic biquandle*.

(2) [2] Let k be a commutative ring with unity 1 and with units s and t such that $(1 - s)(1 - t) = 0$. Then the function $R : k \times k \to k \times k$ given by

$$R(a, b) = (R_1(a, b), R_2(a, b)) = ((1 - s)a + sb, ta + (1 - t)b)$$

is a set-theoretic Yang-Baxter operator, and (k, R_1, R_2) forms a biquandle, called an *Alexander biquandle*.
For example, let $k = \mathbb{Z}_m$ with units s and t such that $m = |(1-s)(1-t)|$, then the function R defined as above forms a set-theoretic Yang-Baxter operator and $\mathbb{Z}_{m;s,t} := (\mathbb{Z}_m, R_1, R_2)$ is a biquandle.

3. Normalized homology of a set-theoretic solution of the Yang-Baxter equation

In this section, we study a normalized homology theory for set-theoretic solutions of the Yang-Baxter equation, defined in a similar way as to obtain the quandle homology [3] from the rack homology [7]. We consturct concrete examples of non-trivial n-cocycles for the Alexander biquandles $\mathbb{Z}_{m;s,t}$.

First, we review the homology theory for the set-theoretic Yang-Baxter equation based on [2]. For a set X, let $R : X \times X \to X \times X$ be a set-theoretic Yang-Baxter operator on X. For each integer $n > 0$, we define the n-chain group $C_n^{YB}(X)$ to be the free abelian group generated by the elements of X^n and the n-boundary homomorphism $\partial_n^{YB} : C_n^{YB}(X) \to C_{n-1}^{YB}(X)$ by $\sum_{i=1}^{n}(-1)^{i+1}(d_{i,n}^l - d_{i,n}^r)$, where the two face maps $d_{i,n}^l, d_{i,n}^r : C_n^{YB}(X) \to C_{n-1}^{YB}(X)$ are given by

$$d_{i,n}^l = (R_2 \times \mathrm{Id}_X^{\times(n-2)}) \circ (\mathrm{Id}_X \times R \times \mathrm{Id}_X^{\times(n-3)}) \circ \cdots \circ (\mathrm{Id}_X^{\times(i-2)} \times R \times \mathrm{Id}_X^{\times(n-i)}),$$

$$d_{i,n}^r = (\mathrm{Id}_X^{\times(n-2)} \times R_1) \circ (\mathrm{Id}_X^{\times(n-3)} \times R \times \mathrm{Id}_X) \circ \cdots \circ (\mathrm{Id}_X^{\times(i-1)} \times R \times \mathrm{Id}_X^{\times(n-i-1)}).$$

Then $C_*^{YB}(X) := (C_n^{YB}(X), \partial_n^{YB})$ forms a chain complex, and the yielded homology $H_*^{YB}(X)$ is called the *set-theoretic Yang-Baxter homology* of X.

Consider the subgroup $C_n^D(X)$ of $C_n^{YB}(X)$ defined by

$$C_n^D(X) = \text{span}\{(x_1, \ldots, x_n) \in C_n^{YB}(X) \mid R(x_i, x_{i+1}) = (x_i, x_{i+1}) \text{ for some } i = 1, \ldots, n-1\},$$

if $n \geq 2$, otherwise we let $C_n^D(X) = 0$.

Proposition 3.1. *$(C_n^D(X), \partial_n^{YB})$ is a sub-chain complex of $(C_n^{YB}(X), \partial_n^{YB})$.*

The homology $H_n^D(X) = H_n(C_*^D(X))$ is called the *degenerate set-theoretic Yang-Baxter homology groups* of X. Consider the quotient chain complex $C_*^{NYB}(X) := (C_n^{NYB}(X), \partial_n^{NYB})$, where $C_n^{NYB}(X) = C_n^{YB}(X)/C_n^D(X)$, and ∂_n^{NYB} is the induced homomorphism. For an abelian group A, define the chain and cochain complexes $C_*^{NYB}(X; A) := (C_n^{NYB}(X; A), \partial_n^{NYB})$ and $C^*_{NYB}(X; A) := (C^n_{NYB}(X; A), \delta^n_{NYB})$, where

$$C_n^{NYB}(X; A) = C_n^{NYB}(X) \otimes A, \ \ \partial_n^{NYB} = \partial_n^{NYB} \otimes \text{Id}_A,$$

$$C^n_{NYB}(X; A) = \text{Hom}(C_n^{NYB}(X), A), \ \ \delta^n_{NYB} = \text{Hom}(\partial_n^{NYB}, \text{Id}_A).$$

Definition 3.2. Let R be a set-theoretic Yang-Baxter operator on X. For a given abelian group A, the homology group and cohomology group

$$H_n^{NYB}(X; A) = H_n(C_*^{NYB}(X; A)) = Z_n^{NYB}(X; A)/B_n^{NYB}(X; A),$$

$$H^n_{NYB}(X; A) = H^n(C^*_{NYB}(X; A)) = Z^n_{NYB}(X; A)/B^n_{NYB}(X; A)$$

are called the nth *normalized set-theoretic Yang-Baxter homology group of X with coefficient group A* and the nth *normalized set-theoretic Yang-Baxter cohomology group of X with coefficient group A.*

Lemma 3.3. *For an Alexander biquandle X, the face maps $d^l_{i,n}, d^r_{i,n} : C_n^{YB}(X) \to C_{n-1}^{YB}(X)$ have the formulas:*

$$d^l_{i,n}(x_1, \ldots, x_n) = (tx_1 + (1-t)x_i, \ldots, tx_{i-1} + (1-t)x_i, x_{i+1}, \ldots, x_n),$$

$$d^r_{i,n}(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, (1-s)x_i + sx_{i+1}, \ldots, (1-s)x_i + sx_n).$$

Theorem 3.4. *Let $X = \mathbb{Z}_{m;s,t}$ be an Alexander biquandle. For $n \geq 2$, the map $\theta_n \in C^n_{NYB}(X; \mathbb{Z}_m)$ defined by*

$$\theta_n(x_1, \ldots, x_n) = \prod_{i=1}^{n-1}(x_i - x_{i+1})$$

and extending linearly to all elements of $C_n^{NYB}(X)$ is an n-cocycle.

4. Classifying space of a biquandle

For a given set X, let $R = (R_1, R_2) : X \times X \to X \times X$ be a set-theoretic Yang-Baxter operator. We define the face maps $d_i^r, d_i^l : X^n \to X^{n-1}$ by

$$d_i^r = (\text{Id}_X^{\times(n-2)} \times R_1) \circ (\text{Id}_X^{\times(n-3)} \times R \times \text{Id}_X) \circ \cdots \circ (\text{Id}_X^{\times(i-1)} \times R \times \text{Id}_X^{\times(n-i-1)}),$$

$$d_i^l = (R_2 \times \text{Id}_X^{\times(n-2)}) \circ (\text{Id}_X \times R \times \text{Id}_X^{\times(n-3)}) \circ \cdots \circ (\text{Id}_X^{\times(i-2)} \times R \times \text{Id}_X^{\times(n-i)}).$$

Then $\mathcal{X} = (X^n, d_i^r, d_i^l)$ forms a pre-cubical set, where X^0 is a singleton set $\{*\}$. In this case the homology of its geometric realization $|\mathcal{X}|$ is the homology for the set-theoretic Yang-Baxter equation in [2].

When (X, R_1, R_2) is a birack, its geometric realization is called a *birack space*. If $(X, R_1.R_2)$ is a biquandle, the birack space can be transformed into a more interesting space, called a *biquandle space*, which can be used for constructing link invariants, in analogy to the way quandle spaces in [11] where obtained from rack spaces [7].

We inductively define the n-skeleton $(n \geq 3)$ of a biquandle space.
Let (X, R_1, R_2) be a biquandle, and let $|\mathcal{X}|_n$ be the n-skeleton of $|\mathcal{X}|$. For each $x \in X$, we denote the unique element $y \in X$ such that $R(x, y) = (x, y)$ by $\overline{x}$, i.e., $R(x, \overline{x}) = (x, \overline{x})$. Consider the subset $D^m = \{(x_1, \ldots, x_m) \in X^m \mid x_{i+1} = \overline{x_i} \text{ for some } 1 \leq i \leq m-1\}$ of X^m.

Note that the rectangle labeled by $\mathbf{x} \in D^2$ forms the wedge sum of one 2-sphere S^2 and some circles (more precisely, it is either $S^2 \vee S^1$ or $S^2 \vee S^1 \vee S^1$ in $|\mathcal{X}|_n$. For each $\mathbf{x} \in D^2$, we attach a 3-cell B^3 to the 2-sphere S^2 via a homeomorphism $\partial(B^3) \to S^2$. Denote the resulting space by BX_3.

Let $\mathbf{x} \in D^3$, and let $D^2_{\mathbf{x}} = \{d_i^{\varepsilon}(\mathbf{x}) \mid 1 \leq i \leq 3,\ \varepsilon \in \{l, r\}\} \cap D^2$. The cube labeled by $\mathbf{x} \in D^3$ together with the 3-cells attached to each $\mathbf{y} \in D^2_{\mathbf{x}}$ in the previous step forms the wedge sum of one 3-sphere S^3 and some 2-spheres and circles in BX_3. For each $\mathbf{x} \in D^3$, we attach a 4-cell B^4 to the 3-sphere S^3 via a homeomorphism $\partial(B^4) \to S^3$. The resulting space, denoted by BX_4, is the 4-skeleton of the biquandle space of X.

Suppose that BX_{n-1} is the cell complex obtained in the previous step. For each $\mathbf{x} \in D^{n-1}$, we can build the wedge sum of one $(n-1)$-sphere S^{n-1} and some p-spheres $(p < n-1)$ in BX_{n-1} similar to the above by inductively attaching k-dimensional cells for $3 \leq k \leq n-1$ with respect to the elements of $D^{k-1}_{\mathbf{x}} = \{d_{i_{n-k}}^{\varepsilon_{n-k}} \circ \cdots \circ d_{i_1}^{\varepsilon_1}(\mathbf{x}) \mid 1 \leq i_j \leq n-j,\ \varepsilon_j \in \{l, r\},\ 1 \leq j \leq n-k\} \cap D^{k-1}$. Again, we attach a n-cell B^n to the $(n-1)$-sphere S^{n-1} via a homeomorphism $\partial(B^n) \to S^{n-1}$ in order to construct the n-skeleton BX_n of the *biquandle space* BX. By construction, the homology of the biquandle space BX coincides with the normalized set-theoretic Yang-Baxter homology of the biquandle X.

The 4-skeleton BX_4 of the biquandle space is especially important for classical and surface-knot-theoretic applications.

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VARIATION OF KÄHLER-EINSTEIN METRICS ON BOUNDED STRONGLY PSEUDOCONVEX DOMAINS IN KÄHLER MANIFOLDS

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Abstract. In this survey, we introduce the variation of the Kähler-Einstein metrics on a family of bounded pseudoconvex domains in Kähler manifolds and its positivity.

1. Introduction

On a family of canonically polarized compact Kähler manifolds, the variation of the Kähler-Einstein metrics is represented by a curvature form of the relative canonical line bundle. In [5], Schumacher has proved that the variation of Kähler-Einstein metrics on a family of canonically polarized compact Kähler manifolds is semi-postive. He has also proved that it is strictly positive if the family is effectively parametrized. This celebrated theorem implies many important applications on the moduli space of canonically polarized compact Kähler manifolds, especially the extension of curvature forms and line bundles. Moreover, it also gives a nice curvature formula of the Weil-Petersson metric on the moduli space ([5]).

In this survey article, we will explain the brief idea of the Schumacher's theorem and how to apply his method to a *family of bounded strongly pseudoconvex domains in Kähler manifolds*.

2. Preliminaries

Let $p : X^{n+d} \to Y^d$ be a smooth family of Kähler manifolds, i.e., p is a surjective holomorphic submersion. Taking a local coordinate $(s^1, \ldots, s^d)$ of Y and a local coordinate $(z^1, \ldots, z^n)$ of a fiber of p, $(z^1, \ldots, z^n, s^1, \ldots, s^d)$ forms a local coordinate of X such that under this coordinate, the holomorphic mapping p is given by

$$p(z^1, \ldots, z^n, s^1, \ldots, s^d) = (s^1, \ldots, s^d).$$

We call this an *admissible coordinate of* p.

2.1. Horizontal lifts and geodesic curvatures. For a complex manifold M, we denote by $T'M$ the complex tangent bundle of type $(1,0)$.

Definition 2.1. Let $V \in T'Y$ and τ be a real $(1,1)$-form on X. Suppose that τ is positive-definite on each fiber X_y.

1. A vector field V_τ of type $(1,0)$ is called a *horizontal lift* of V if V_τ satisfies the following:
   (i) $\langle V_\tau, W \rangle_\tau = 0$ for all $W \in T'X_y$,
   (ii) $dp(V_\tau) = V$.
2. The *geodesic curvature* $c(\tau)(V)$ of τ along V is defined by the norm of V_τ with respect to the sesquilinear form $\langle \cdot, \cdot \rangle_\tau$ induced by τ, namely,

$$c(\tau)(V) = \langle V_\tau, V_\tau \rangle_\tau .$$

Suppose that Y is 1-dimensional. Then it is well known that

$$\tau^{n+1} = c(\tau) \cdot \tau^n \wedge \sqrt{-1} ds \wedge d\bar{s}. \tag{6.3}$$

It follows that if $c(\tau) > 0$ (resp. ≥ 0), then τ is a positive (resp. semi-positive) real $(1,1)$-form as τ is positive-definite when restricted to X_s.

2.2. Families of canonically polarized compact Kähler manifolds. Let $p : X^{n+d} \to Y^d$ be a smooth family of canonically polarized compact Kähler manifolds, i.e., p is a surjective holomorphic submersion such that on each fiber $X_y := p^{-1}(y)$, the canonical line bundle K_{X_y} of X_y is positive. Then Yau's theorem implies that there exists a Kähler-Einstein metric ω_y^{KE} with Ricci curvature -1 on each fiber X_y ([1, 7]). The variation of the Kähler-Einstein metric ρ of the family of canonically polarized compact Kähler manifolds is defined by

$$\rho = i\partial\bar{\partial} \log((\omega_y^{KE})^n \wedge p^*(dV_s)),$$

Then the Kähler-Einstein condition immediately implies that

$$\rho|_{X_y} = \omega_y^{KE}.$$

Schumacher proves that the geodesic curvature of ρ satisfies an elliptic partial differential equation.

Theorem 2.2 (Schumacher [5]). *The geodesic curvature $c(\rho)$ satisfies that*

$$-\Delta c(\rho) + c(\rho) = \left|\bar{\partial} v_\rho\right|^2 \tag{6.4}$$

on each fiber X_y, where v_ρ is the horizontal lift of $v := \partial/\partial s$ with respect to ρ.

Applying the maximum principle to (6.6), one can easily see that $c(\rho)$ is nonnegative. This implies that ρ is semi-positive on X. Moreover, if $\bar{\partial} v_\rho$ is not identically vanishes, then $c(\rho)$ is positive by the property of the solution of (6.6) or the heat kernel estimate. (For the detail, see [5].) It is remarkable that $\bar{\partial} v_\rho$ is the harmonic representative of the Kodaira-Spencer class of $p : X \to Y$. Therefore, if the family is not locally trivial then ρ is positive-definite on X. Therefore, we have the following theorem.

Theorem 2.3 (Schumacher [5]). *Let $p : X \to Y$ be a family of canonically polarized compact Kähler manifolds. Then the variation of the Kähler-Einstein metrics ρ is positive semi-definite. Moreover, ρ is strictly positive-definite if the family is not locally trivial.*

3. Families of bounded pseudoconvex domains

In this section, we will apply Schumacher's method to families of strongly pseudoconvex domains. Before going to the families of strongly pseudoconvex domains, we recall the complete Kähler-Einstein metric on a strongly pseudoconvex domain.

3.1. Kähler-Einstein metric on a strongly pseudoconvex domain. Let Ω be a smooth bounded strongly pseudoconvex domain in a Kähler manifold (M, ω) such that $\mathrm{Ric}(\omega) < 0$. This gives us a new Kähler form ω^0 on M, defined by

$$\omega^0 := -\frac{1}{n+1} \mathrm{Ric}(\omega)$$

where n is the complex dimension of M. Let r be a defining function of Ω which is strictly plurisubharmonic on a neighborhood of $\partial\Omega$. Then $-\log(-r)$ is strictly plurisubharmonic near $\partial\Omega$. It is easy to see that

$$\omega_r^0 := \omega^0 - i\partial\bar{\partial} \log(-r)$$

is a complete Kähler metric on Ω. Moreover, (Ω, ω_r^0) has bounded geometry of infinite order (see Proposition 1.3 in [2]). The following theorem due to Cheng and Yau gives a solution of the complex Monge-Ampère equation.

Theorem 3.1 (Cheng, Yau [2]). *If $F \in C^\infty(\overline{\Omega})$, then there exists a solution ϕ of the equation:*

$$(\omega_r^0 + i\partial\overline{\partial}\phi)^n = e^{(n+1)\phi + F}(\omega_r^0)^n, \tag{6.5}$$

which is called a complex Monge-Ampère equation.

Applying Theorem 3.1 with $F := \log\left[(-r)^{-(n+1)}\frac{(\omega)^n}{(\omega_r^0)^n}\right] \in C^\infty(\overline{\Omega})$, the complex Monge-Ampère equation implies that

$$\mathrm{Ric}(\omega_r^0 + i\partial\overline{\partial}\phi) = -(n+1)(\omega_r^0 + i\partial\overline{\partial}\phi).$$

The uniqueness of the Kähler-Einstein metric ω^{KE} on Ω says that

$$\omega^{KE} = \omega_r^0 + i\partial\overline{\partial}\phi.$$

3.2. Families of strongly pseudoconvex domains. Let $p : X \to Y$ be a surjective holomorphic submersion, where X and Y are complex manifolds and let D be a bounded smooth domain in X such that every fiber $D_y := D \cap p^{-1}(y)$ with $y \in Y$ is a bounded strongly pseudoconvex domain in $X_y := p^{-1}(y)$. We call $p : X \to Y$ with D a *holomorphic family of bounded strongly pseudoconvex domains* in X.

If there exists a Kähler form ω on X and satisfies that the Ricci curvature $\mathrm{Ric}(\omega_y)$ of $\omega_y := \omega|_{X_y}$ is negatively curved for every $y \in Y$, then Theorem 3.1 implies that there exists a unique complete Kähler metric ω_y^{KE} on D_y satisfying

$$\mathrm{Ric}(\omega_y^{KE}) = -(n+1)\omega_y^{KE},$$

where n is the dimension of D_y (cf. [2]). As before, we define the variation of Kähler-Einstein metrics by

$$\rho := \frac{1}{n+1}\Theta_{h_{D/Y}} = \frac{1}{n+1} i\partial\overline{\partial}\log((\omega_y^{KE})^n \wedge p^*(dV_s)),$$

The Kähler-Einstein condition implies that

$$\rho|_{D_y} = \frac{1}{n+1}\Theta_{h_{D/Y}}|_{D_y} = \omega_y^{KE},$$

for all $y \in Y$. Since the proof of Theorem 2.2 is local, we have the same PDE for $c(\rho)$.

Proposition 3.2. *The geodesic curvature $c(\rho)$ satisfies that*

$$-\Delta c(\rho) + (n+1)c(\rho) = \left|\overline{\partial}v_\rho\right|^2 \tag{6.6}$$

on each fiber D_y, where v_ρ is the horizontal lift of $v := \partial/\partial s$ with respect to ρ.

Unlike the previous case, now our fiber X_y is not compact. This means that we cannot apply the maximum principle directly. However, if the geodesic curvature $c(\rho)$ is bounded from below, then the Omori-Yau almost maximum principle ([6]) implies that there exists a sequence $\{x_k\} \subset D_y$ such that

(i) $\inf_{x \in D_y} c(\rho)(x) = \lim_{k\to\infty} c(\rho)(x_k)$,

(ii) $\lim_{k\to\infty} \nabla c(\rho)(x_k) = 0$, and $\liminf_{k\to\infty} \Delta c(\rho)(x_k) \geq 0$.

It follows from Proposition (6.6) that

$$(n+1)c(\rho)(x_k) = \left|\overline{\partial}v_\rho\right|^2 + \Delta c(\rho)(x_k) \geq 0.$$

Taking $k \to \infty$, we have $c(\rho) \geq 0$. Since the Kähler-Einstein metric is real-analytic, $c(\rho)$ and v_ρ are also real-analytic. Therefore, we can apply the following proposition.

Proposition 3.3 (cf. [5, 3]). *Let u and f be real-analytic, non-negative, real function on a neighborhood $U \subset \mathbb{C}^n$ of 0. Let ω_U be a real-analytic Kähler form on U and C be a positive constant. Suppose*

$$-\Delta_{\omega_U} u + Cu = f$$

holds. If $u(0) = 0$, then both u and f are vanish identically in a neighborhood of 0.

The above proposition with $u := c(\rho)$ implies that $c(\rho) \equiv 0$ or $c(\rho) > 0$. Now the positivity of ρ follows from the following:

PROPOSITION 3.4. *For each fiber D_y,*

$$c(\rho)(x) \to \infty \quad \textit{as} \quad x \to \partial D_y, \tag{6.7}$$

provided D is a strongly pseudoconvex domain in X.

This proposition comes from the boundary behavior of the variation of the solutions of complex Monge-Ampère equations. (For the detail, see [4].) Therefore we have the following theorem.

THEOREM 3.5 (Choi-Yoo [4]). *Let $p : X \to S$ with $D \subset X$ be a holomorphic family of strongly pseudoconvex domains in X. Suppose that there exists a Kähler matric ω such that*

$$\mathrm{Ric}(\omega_y) < 0$$

on every fiber X_y. If the total space D is strongly pseudoconvex, then ρ is positive-definite on D'.

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MINIMAL SURFACES AND SOLITONS FOR THE MEAN CURVATURE FLOW

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Abstract. Minimal surfaces are solutions to a variational problem for the area functional. Minimal surfaces theory is related to several mathematical disciplines. We introduce interesting examples and some properties of them. And self-similar solutions and translating solitons are not only special solutions of mean curvature flow (MCF) but a key role in the study of singularities of MCF. They have received a lot of attention. We introduce some examples of self-similar solutions and translating solitons for the MCF and give rigidity results of some of them.

1. Minimal surfaces

In mathematics, a minimal surface is a surface that locally minimizes its area. This is equivalent to having zero mean curvature. The term *minimal surface* is used because these surfaces originally arose as surfaces that minimized total surface area subject to some constraint. Physical models of area-minimizing minimal surfaces can be made by dipping a wire frame into a soap solution, forming a soap film, which is a minimal surface whose boundary is the wire frame. However, the term is used for more general surfaces that may self-intersect or do not have constraints. For a given constraint there may also exist several minimal surfaces with different areas (for example, see minimal surface of revolution): the standard definitions only relate to a local optimum, not a global optimum.

Minimal surfaces can be defined in several equivalent ways in $\mathbb{R}^3$. The fact that they are equivalent serves to demonstrate how minimal surface theory lies at the crossroads of several mathematical disciplines, especially differential geometry, calculus of variations, potential theory, complex analysis and mathematical physics.

Local least area definition: A surface $M \subset \mathbb{R}^3$ is minimal if and only if every point $p \in M$ has a neighborhood, bounded by a simple closed curve, which has the least area among all surfaces having the same boundary. This property is local: there might exist regions in a minimal surface, together with other surfaces of smaller area which have the same boundary. This property establishes a connection with soap films; a soap film deformed to have a wire frame as boundary will minimize area.

Variational definition: A surface $M \subset \mathbb{R}^3$ is minimal if and only if it is a critical point of the area functional for all compactly supported variations. This definition makes minimal surfaces a 2-dimensional analogue to geodesics, which are analogously defined as critical points of the length functional.

Minimal surface curvature planes. On a minimal surface, the curvature along the principal curvature planes are equal and opposite at every point. This makes the mean curvature zero. Mean curvature definition: A surface $M \subset \mathbb{R}^3$ is minimal if and only if its mean curvature is equal to zero at all points. A direct implication of this definition is that every point on the surface is a saddle point with equal and opposite principal curvatures. Additionally, this makes minimal surfaces into the static solutions of mean curvature flow. By the Young–Laplace equation, the mean curvature of a soap film is proportional to the difference in pressure between the sides. If the soap film does not enclose a region, then this will make its mean curvature zero. By contrast, a spherical soap bubble encloses a region which has a different pressure from the exterior region, and as such does not have zero mean curvature.

Differential equation definition: A surface $M \subset \mathbb{R}^3$ is minimal if and only if it can be locally expressed as the graph of a solution of $div\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)$. The partial differential equation in this definition was originally found in 1762 by Lagrange, and Jean Baptiste Meusnier discovered in 1776 that it implied a vanishing mean curvature.

2. solitons for the mean curvature flow

A smooth family of immersions $F : \Sigma \times [0,T) \to \mathbb{R}^{n+1}$ is a solution of the *mean curvature flow* (MCF) if F satisfies the following parabolic equation:

$$\frac{\partial}{\partial t}F(p,t) = \overrightarrow{H}(p,t), \tag{6.8}$$

for all $(p,t) \in \Sigma \times [0,T)$, where $\overrightarrow{H}$ is the mean curvature vector. The MCF is the negative gradient flow of the area functional. It is well-known that any closed hypersurface occurs singularities in finite time under the MCF. Thus, it is important to study singularities of the MCF. Huisken [4] and Huisken and Sinestrari [5] showed that there are two types of singularities: type-I and type-II that are represented by self-shrinkers and translating solitons, respectively. An n-dimensional submanifold in $\mathbb{R}^{n+m}$ is called a *self-shrinker* if it satisfies

$$\vec{H} = -x^{\perp},$$

where $\vec{H}$ and $x^{\perp}$ denote the mean curvature vector and the normal part of the position vector x to the submanifold. A *translating soliton* for the MCF is a submanifold in $\mathbb{R}^{n+m}$ satisfying the following equation:

$$\vec{H} = \mathrm{v}^{\perp},$$

where $\mathrm{v}^{\perp}$ denotes the normal part of a unit constant vector v to the submanifold. A translating soliton is not only a blow-up limit flow of type-II singularity, but also a special solution that moves only in a constant direction v without deforming its shape under the MCF, namely, the solution is as follows:

$$F(p,t) = F(p) + \mathrm{v}t,$$

where $F(p) = F(0,p)$.

3. examples of translating solitons

Example 3.1 (Product of minimal submanifold). The simplest translating soliton is a plane parallel to v in $\mathbb{R}^3$ as a product of a line and $\mathbb{R}$ parallel to the direction v. From this perspective, the translating solitons in $\mathbb{R}^{n+1}$ can be constructed as the product of an $(n-1)$-dimensional minimal submanifold M and $\mathbb{R}$ parallel to the direction v, *i.e.*, $M \times \mathbb{R}$. There are numerous of translating solitons arising from minimal submanifolds. In [8], Nadirashvili constructed a complete, non-proper, minimal disk in the unit ball. Specifically, a complete non-proper translating soliton can be obtained from Nadirashvili's minimal surface.

Example 3.2 (Grim reaper cylinders). The grim reaper $y = -\log\cos(x)$ is a translating soliton on $\mathbb{R}^2$, *i.e.*, the only eternal solution of the MCF in $\mathbb{R}^2$, which is also known as the curve-shortening flow. Its product surface, which is a cylindrical surface of the grim reaper, is called a *canonical grim reaper cylinder* whose suitable combination of rotation and dilation is called a *grim reaper cylinder*. The following parametrization is for a family of grim reaper cylinders:

$$X_\theta(u,v) = \left(s, t, -\frac{1}{\cos^2(\theta)}\log\cos(s\cos(\theta)) + t\tan(\theta)\right).$$

In particular, the grim reaper cylinder is a one-parameter family of cylindrical surfaces from the canonical grim reaper cylinder to the plane parallel to $\mathrm{v} = e_3$.

EXAMPLE 3.3 (translating bowl and winglike translator). Altschuler and Wu [1] and Clutterbuck, Schnürer and Schulze [2] showed the existence of the translating bowl and the winglike translator. These are rotationally symmetric translating solitons and can thus be represented as an immersion $X : I \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$ parametrized by

$$X(s, \phi_1, \cdots, \phi_{n-1}) = (x(s)\Phi(\phi_1, \cdots, \phi_{n-1}), y(s)),$$

where Φ is an orthogonal parametrization of the $(n-1)$-dimensional unit sphere. The profile curve $\gamma(s) = (x(s), y(s))$ parametrized by arc-length satisfies the following differential equation:

$$x'(1-n) + nyy' + y(x'y'' - x''y') = 0.$$

In particular, the translating bowl and winglike translator have an asymptotic behavior of $y = x^2$. The Kim and the author [6] rediscovered their asymptotic behaviors of the profile curve using the phase-plane method to the above differential equation.

EXAMPLE 3.4 (Generalized winglike translator). As a generalization of the winglike translator, Kunikawa [7] constructed an m-dimensional translating soliton in $\mathbb{R}^n$. Let N be any minimal submanifold in $\mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$ and $r : [0, \infty] \to \mathbb{R}$ be a function satisfying

$$r'' = (1 + r'^2)\left(1 - \frac{(m-1)r'}{t}\right),$$

which is an m-dimensional winglike translator equation. The immersion $F : M \to \mathbb{R}^n$ defined by $F(t, p) = (tp, r(t))$ where $p \in N$ and $t \in [0, \infty]$ is an m-dimensional translating soliton with the velocity $e_n \in \mathbb{R}^n$.

EXAMPLE 3.5 (Helicoidal translating solitons). Halldorsson [3] proved the existence of the helicoidal rotating solitons under the MCF, which are also known as the helicoidal translating solitons. The authors [6] completely classified the profile curves and analyzed their asymptotic behaviors in the same way as those of the translating bowl or winglike translator. Consider a helicoidal translating soliton Σ with the pitch h whose helicoidal axis is the z-axis. We can parametrize Σ as $X : \Sigma \to \mathbb{R}^3$ by

$$X(s, t) = (x(s)\cos(t), x(s)\sin(t), y(s) + ht),$$

such that the profile curve (x, y) parametrized by arc-length satisfies the following differential equation:

$$(x^2 + 2h^2x'^2)y' + x(h^2 + x^2)(x'y'' - y'x'') - 2xx'(x^2 + h^2x'^2) = 0.$$

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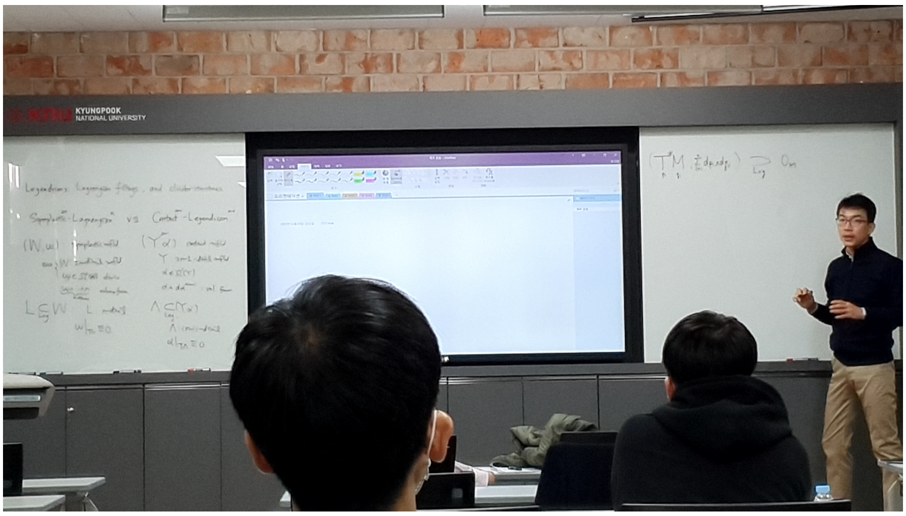
Commemorative Photographs


Real hypersurface in $G_2(C^{m+2})$
A_T
Real hypersurface M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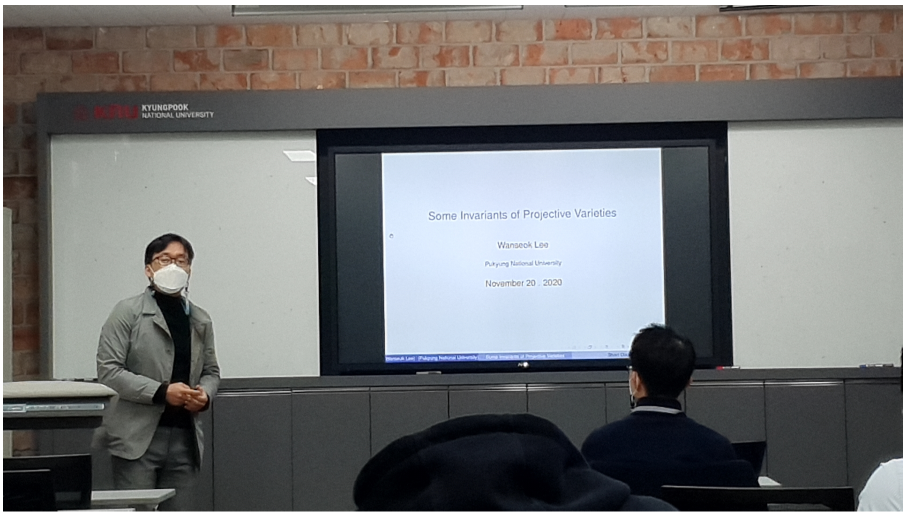
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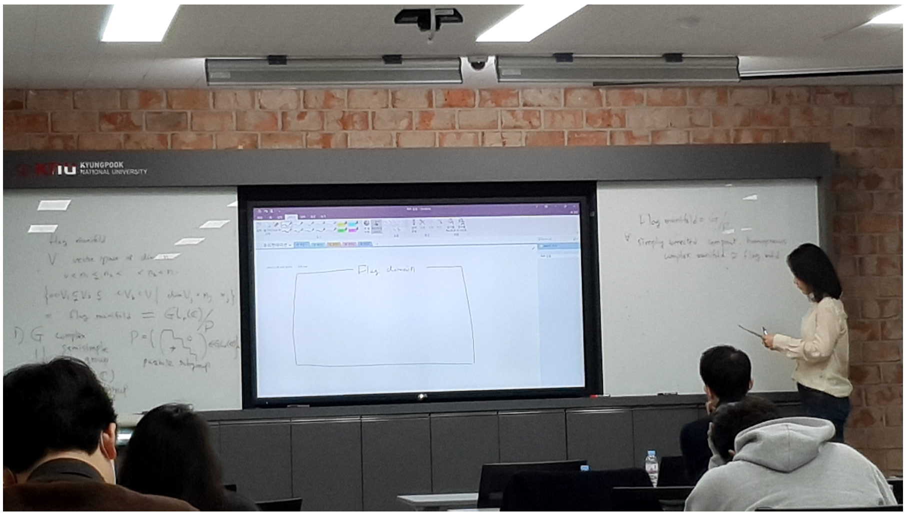
Some Invariants of Projective Varieties
Wanseok Lee
Pukyong National University
November 20 , 2020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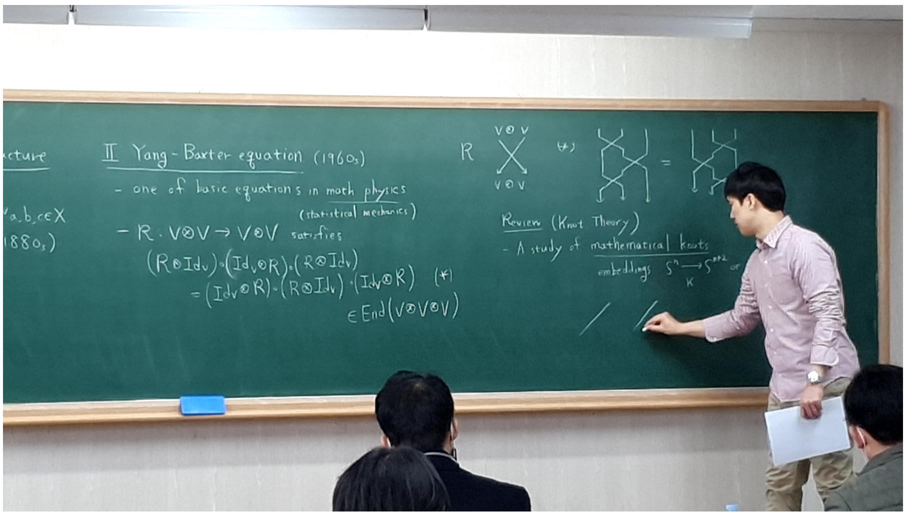
Flag domain


II Yang-Baxter equation (1960s)
- one of basic equations in math physics
(statistical mechanics)
- R : V⊗V → V⊗V satisfies
(R⊗Id_V)∘(Id_V⊗R)∘(R⊗Id_V)
= (Id_V⊗R)∘(R⊗Id_V)∘(Id_V⊗R) (*)
∈ End(V⊗V⊗V)
R
V⊗V
V⊗V
(*)
Review (Knot Theory)
- A study of mathematical knots
embeddings


Fiberwise Kähler-Ricci flows
$p : X \to \mathbb{D}$ a family of complex (Kähler) manifolds.
Suppose ω is a Kähler metric on X.
Then $(X_s, \omega|_{X_s})$ is a Kähler manifold for every $s \in \mathbb{D}$
Then one can consider the following Kähler-Ricci flow on each fiber X_s.
$\frac{\partial}{\partial t}\omega_s(t) = -\mathrm{Ric}(\omega_s(t)) - \lambda\omega_s(t)$.
$\omega_s(0) = \omega|_{X_s}$.
Suppose that the Kähler-Ricci flow has long time solution for each fiber X_s. Then this induces a real (1,1)-form $\omega(t)$ on X which satisfies
$\omega(0) = \omega$.
$\Rightarrow \omega(t)$ is called the fiberwise Kähler-Ricci flow.


Glancing the surface theory
$\Sigma \subset \mathbb{R}^3$ is a surface.
$\Longleftrightarrow X : (u, v) \in M \to \mathbb{R}^3$ and $X(M) = \Sigma$
$E = X_u \cdot X_u$, $F = X_u \cdot X_v$, $G = X_v \cdot X_v$
$\mathrm{Area}(\Sigma) = \int_M \sqrt{EG - F^2}\,du\,dv = \int_M dA$
With $N = \frac{X_u \times X_v}{|X_u \times X_v|}$, $l = X_{uu} \cdot N$, $m = X_{uv} \cdot N$, $n = X_{vv} \cdot N$
Mean curvature $H = \frac{En + Gl - 2Fm}{2(EG - F^2)}$.
Σ is a minimal surface if $H \equiv 0$
Σ is a constant mean curvature (cm
$H \equiv C(\neq 0)$