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Lect I. Affine toric varieties

1 Affine varieties

called "affine var."

- $I \subseteq S := \mathbb{C}[x_1, \cdots, x_n] \rightsquigarrow V(I) := \{p \in \mathbb{C}^n : f(p) = 0 \ \forall f \in I\}$
- $V \subseteq \mathbb{C}^n \rightsquigarrow I(V) := \{f \in S : f|_V \equiv 0\}$ ← defining ideal of V

* $I(V)$: radical ideal. "Hilbert Nullstellensatz" : $\{$radical ideals$\}$ $\xleftrightarrow{1:1}$ $\{$aff. var.'s$\}$ order-reversing

- $\mathbb{C}[V] := S/_{I(V)}$ ← coord. ring of V

$\quad \ni$

$\quad f$: regular ftn (= polynomial ftn) on V

Prop. i)$_p$ $\mathbb{C}[V]$: int. domain $\Leftrightarrow$ $I(V)$: prime $\Leftrightarrow$ V : irred.

+ Defn ii)$_d$ $\phi : V_1 \to V_2$ is called morphism if it is induced from

$$\phi^* : \mathbb{C}[V_2] \to \mathbb{C}[V_1]$$

iii)$_p$ Two var.'s are iso. iff $\mathbb{C}[V_1] \simeq \mathbb{C}[V_2]$

iv)$_p$ $p \in V$ gives $m_p := \{f \in \mathbb{C}[V] : f(p) = 0\} \subseteq \mathbb{C}[V]$

↳ maximal ideal

All max. of $\mathbb{C}[V]$ arise in this way

Hw: Given $\phi^* : \mathbb{C}[y_1 \cdots y_m]/_J \to \mathbb{C}[x_1, \cdots, x_n]/_I$, define $\phi : V_1 \to V_2$

$$\ulcorner \quad \phi := (\underbrace{\phi^*(\bar{y}_1)}, \cdots, \underbrace{\phi^*(\bar{y}_m)}) \quad \lrcorner$$

in $\bar{x}_1, \cdots, \bar{x}_n$

Prove that ϕ : well-defined.

: point $\rightsquigarrow \mathbb{C}[pt] = \mathbb{C}$

$\therefore p \to V \leftrightarrow \mathbb{C}[x_1 \cdots x_n]/_I \to \mathbb{C}$. with $\ker = \langle x_i - a_i \rangle$

$\bar{x}_i \mapsto a_i$

Lemma. R : coord. ring of aff. var. iff it is ~~fin. gen.~~ $\mathbb{C}$-alg. with no nilpotent.

KAIST A4 FINE NOTE

2 Zariski topology

- Affine var V has two topol. : $\begin{cases} \text{classical top.} \leftarrow \text{induced top of std top on } \mathbb{C}^n \\ \text{Zariski top.} \end{cases}$

$W \subseteq V$ is closed iff $W = \{ p \in V : f(p) = 0 \text{ for } {}^{\forall} f \in J \} \subseteq V$ for some $J \subseteq \mathbb{C}[V]$

* Equivalently, $\exists$ ideal J of $\mathbb{C}[V]$ s.t $W = V(J)$

- For $S \subseteq V$, $\bar{S}$: Zariski closure = smallest subvar. of V containing S

Prop (HW) For $f \in \mathbb{C}[V]$, let $V_f := \{ p \in V : f(p) \neq 0 \} = V \setminus V(f)$ ← obviously open

Then $\{V_f\}_{f \in \mathbb{C}[V]}$ forms a basis of the Zariski top.

- $V_f \cap V_g = V_{fg}$ ...
- $V_f \cap V_g = V_{\langle f, g \rangle}$

- We can regard V_f as an affine variety in $\mathbb{C}^n \times \mathbb{C}$ s.t

$V_f = \{ (p, z) \in \mathbb{C}^n \times \mathbb{C} : f(p) \cdot z = 1 \}$ (e.g. $\mathbb{C}^* = V_z$)

Prop. $\mathbb{C}[V_f] \simeq \mathbb{C}[V]_f := \left\{ \frac{g}{f^l} \in \mathbb{C}(V) : g \in \mathbb{C}[V],\ l \geq 0 \right\}$

(inverting f)

Ex. $\underbrace{(\mathbb{C}^*)^n}_{V} \subseteq \mathbb{C}^n$

$= \mathbb{C}^n \setminus V(x_1 \cdots x_n)$ & $\mathbb{C}[V] = \mathbb{C}[x_1, \cdots, x_n]_{x_1 \cdots x_n} = \mathbb{C}[x_1^{\pm 1} \cdots x_n^{\pm 1}]$

$= \mathbb{C}^n \setminus V(x_1) \cup \cdots \cup V(x_n)$

$= \bigcap_i \mathbb{C}[V]_{x_i}$

3 Normality

Defn Let R: integral domain. R is normal or integrally closed if an elt in $Q(R)$ is integral over R, then it is in R.

An affine var. V is normal if $\mathbb{C}[V]$ is normal.

Ex. Any UFD is normal. (proof?)

Cor. $\mathbb{C}^n$ is normal ($\because \mathbb{C}[x_1 \cdots x_n]$ UFD)

HW. Any smooth irred. affine variety is normal

↳ i) $\mathbb{C}[V] = \bigcap_{p \in V} \mathcal{O}_{V,p}$, $\mathcal{O}_{V,p} := \{ \frac{f}{g} \in \mathbb{C}(V) : g(p) \neq 0 \}$

$= \mathbb{C}[V]_S$, $S = \{ g \in \mathbb{C}[V] : g(p) \neq 0 \}$

ii) If $\{R_\alpha\}_{\alpha \in I}$: family of normal domains s.t $Q(R_\alpha) = K$ $\forall \alpha \in I$.
then $\cap R_\alpha$ is normal.

iii) $\mathcal{O}_{V,p}$ is normal (Indeed UFD)

regular local ↗ Auslander-Buchsbaum, Nagata

Ex. (non-normal var.)

$C := V(x^3 - y^2) \subseteq \mathbb{C}^2$

$\mathbb{C}[C] = \mathbb{C}[x,y]/\langle x^3 - y^2 \rangle$ $(=: R)$ $\rightsquigarrow$ $\frac{\bar{y}}{\bar{x}} \in \mathbb{C}(C)$ which is integral over R ($\because \bar{x} = (\frac{\bar{y}}{\bar{x}})^2$)

But, $\frac{\bar{y}}{\bar{x}} \notin R$ (why?)

$\therefore$ not normal.

Thm. If V is normal, then $\dim V_{sing} \geq 2$.

$\therefore$ normal curve $\Rightarrow$ smooth

" surface $\Rightarrow$ smooth or isolated sing.

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Defn Suppose V is NOT normal. Define

$$\mathbb{C}[V]' := \{ \alpha \in \mathbb{C}(V) : \alpha \text{ is integral over } \mathbb{C}[V] \}$$

(integral closure of $\mathbb{C}[V]$)

Prop $\mathbb{C}[V]'$ is normal & finitely generated.

$\therefore$ $\mathbb{C}[V] \hookrightarrow \mathbb{C}[V]'$ induces $V' \to V$

normalization of V

Ex. In the previous example, $\mathbb{C}[V][\frac{y}{x}] \cong \mathbb{C}[\frac{y}{x}] \therefore \mathbb{C}$ which gives $\mathbb{C} \to V$

$$t \mapsto (t^2, t^3)$$

$\boxed{4}$ Smoothness

- Recall $\mathcal{O}_{V,p}$: set of rational ftns defined at p.

$\mathcal{O}_{V,p}$ has a maximal ideal $m_{V,p} = \{ \phi \in \mathcal{O}_{V,p} : \phi(p) = 0 \}$

Define

$$T_pV := \mathrm{Hom}_{\mathbb{C}}\left(m_{V,p}/m_{V,p}^2 , \mathbb{C} \right)$$

$\hookrightarrow$ Zariski tangent space

Lemma Let $p \in V \subseteq \mathbb{C}^n$ & $I(V) = \langle f_1, \cdots, f_s \rangle \subseteq \mathbb{C}[x_1, \cdots, x_n]$

Then $T_pV \cong \{ \underbrace{d_p(f_i)}_{\sum_j \frac{\partial f_i}{\partial x_j}(p)\, x_j} = 0 : i = 1, \cdots, n \} \subseteq \mathbb{C}^n$ subspace.

Defn. $p \in V$ is smooth if $\dim T_pV = \dim V$.

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5 Affine toric varieties

Defn. Aff. toric var. V : irred var containing $T_N = (\mathbb{C}^*)^n$ as a Zariski open subset s.t T_N-action on itself extends to V.

* Easy lemma. $S \subset \mathbb{C}^m \circlearrowleft T_N$ s.t S : T_N-inv. Then $\bar{S}$: T_N-inv

Ex. $V(x^3-y^2)$, $V(xy-zw)$
non-normal, normal

· We introduce two lattice : N : one parameter subgroups (lattice)
M : dual of N (character lattice)
$= \mathrm{Hom}(N, \mathbb{Z})$

$$u \in N \leftrightarrow u : \mathbb{C}^* \to T_N, \quad t \mapsto (t^{u_1}, \dots, t^{u_n}) \quad \rightsquigarrow m \circ u(t) = t^{\langle u, m \rangle}$$

$$m \in M \leftrightarrow \chi^m : T_N \to \mathbb{C}^*, \quad t \mapsto t^m := t_1^{m_1} \cdots t_n^{m_n}$$

* Three ways of producing aff. toric var.'s

I. For $\mathcal{A} = \{m_1, \dots, m_s\} \subseteq M$ (finite lattice pts), consider

$$\Phi_{\mathcal{A}} : T_N \to \mathbb{C}^s, \quad t \mapsto (t^{m_1}, \dots, t^{m_s})$$
$t^{m_i} = \chi^{m_i}(t)$

$$Y_{\mathcal{A}} := \overline{\Phi_{\mathcal{A}}(T_N)} \supset T = Y_{\mathcal{A}} \cap (\mathbb{C}^*)^s$$

· $T = Y_{\mathcal{A}} \cap (\mathbb{C}^*)^s$

· $\mathrm{Hom}(T, \mathbb{C}^*) = \mathbb{Z}\mathcal{A}$

Lecture II. Normal toric varieties

1 Affine toric variety

Recall: V is aff. toric iff $T_N := (\mathbb{C}^*)^n \underset{\text{open}}{\hookrightarrow} V$, $T_N \circlearrowleft$, $T_N \circlearrowleft$

• Three ways of producing aff. toric var.

I (finite lattice pts) $\mathcal{A} = \{m_1, \cdots, m_s\} \subseteq M = \{T_N \to \mathbb{C}^*\} \cong \mathbb{Z}^n$

$m \in M$, $t \mapsto t^m$, χ^m

$$\Phi_{\mathcal{A}} : (\mathbb{C}^*)^n \to \mathbb{C}^s$$
$$t \mapsto (t^{m_1}, \cdots, t^{m_s}) \qquad t^{m_i} := \chi^{m_i}(t)$$

$$Y_{\mathcal{A}} := \overline{\Phi_{\mathcal{A}}(T_N)} \supset T = Y_{\mathcal{A}} \cap (\mathbb{C}^*)^s$$
(subgroup of $(\mathbb{C}^*)^s$)

II (toric ideal) Compute $I(Y_{\mathcal{A}}) \subset \mathbb{C}[x_1, \cdots, x_s]$. Consider

$$0 \to L \to \mathbb{Z}^s \xrightarrow{\psi} M \to 0, \quad l = (l_1, \cdots, l_s) \in \underbrace{\ker \psi}_{L}$$
$$e_i \mapsto m_i \qquad \Leftrightarrow \sum_{i=1}^{s} l_i m_i = 0.$$

Note that : $\{(t^{m_1}, \cdots, t^{m_s}) \in \mathbb{C}^s\} \underset{\text{open}}{\subseteq} Y_{\mathcal{A}}$

$\therefore (t^{m_1}, \cdots, t^{m_s})^l = t^{\sum l_i m_i} = 1. \Leftrightarrow l \in \underbrace{\ker \psi}_{L}$

Setting $l = l_+ - l_-$, $l_+, l_- \in \mathbb{N}^s$, we have $x^{l_+} - x^{l_-} \in I(Y_{\mathcal{A}})$

Prop $I(Y_{\mathcal{A}}) = \langle x^{l_+} - x^{l_-} : l \in L \rangle = \langle x^\alpha - x^\beta : \alpha, \beta \in \mathbb{N}^s, \alpha - \beta \in L \rangle$

no common variable. $\therefore$ prime.

called "lattice ideal"

Defn Prime lattice ideal is toric ideal

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II. (toric ideal)

$$0 \to L \to \mathbb{Z}^s \xrightarrow{\varphi} M \to 0 \qquad L := \ker\varphi$$

$$e_i \longmapsto m_i$$

$l = (l_1 \cdots l_s) \rightsquigarrow \sum l_i m_i = 0$

Set $l = l_+ - l_-$, $l_+, l_- \in \mathbb{N}^s$.

Then any pt in Y_A vanishes on $x^{l_+} - x^{l_-}$

$(t^{m_1}, \cdots, t^{m_s}) \rightsquigarrow t^{\langle m, l \rangle} = 1$

$\rightsquigarrow t^{\langle m, l_+ \rangle} = t^{\langle m, l_- \rangle}$

Prop $I(Y_A) = \langle x^{l_+} - x^{l_-} : l \in L \rangle = \langle x^\alpha - x^\beta : \alpha, \beta \in \mathbb{N}^s, \ \alpha - \beta \in L \rangle$

lattice ideal

Defn. prime lattice ideal is toric ideal

HW: How to obtain A from toric ideal?

III. (affine semigroup) $:= \mathbb{N}A$, $|A| < \infty$, $\subset M$

For S, define

$$\mathbb{C}[S] := \left\{ \sum_{m \in S} c_m x^m : \begin{array}{l} c_m \in \mathbb{C} \\ c_m = 0 \text{ all but} \\ \text{fin. many } m\text{'s} \end{array} \right\}$$

w/ $x^m \cdot x^{m'} := x^{m+m'}$.

Ex $\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$

Prop $\mathbb{C}[S]$ is fin. gen $\mathbb{C}$-alg. & int. domain.

Also, $\operatorname{Spec}\mathbb{C}[S]$: aff. toric var with char. lattice $\mathbb{Z}S$

If $S = \mathbb{N}A$, then $\operatorname{Spec}\mathbb{C}[S] = Y_A$.

Hw: How to obtain $\mathcal{A}$ from a given prime lattice ideal?

(Hint: Given I, consider $V(I) \cap (\mathbb{C}^*)^s$: subgroup $\cong T$

$$\downarrow \chi^{m_i} \qquad \text{for } m_i \in \underbrace{\mathrm{Hom}(T, \mathbb{C}^*)}_{M} \;)$$
$$\mathbb{C}^*$$

Prop. I is toric iff it is prime & gen. by binomials.

↳ Proof. Similar to Hw. :)

III (affine semigroups) ⇝ $\underbrace{\text{fin. gen.}}_{\mathcal{A}}$ comm. semigp.

- Assume that $\mathcal{A} \subseteq M$ (some lattice, e.g. $\mathbb{Z}\mathcal{A}$)
- $S = \mathbb{N}\mathcal{A} := \left\{ \sum_{m \in \mathcal{A}} a_m \cdot m \;\middle|\; a_m \in \mathbb{N} \right\}$ lemma. any affine semigp is of this form

Defn. Given S, semigp alg $\mathbb{C}[S] :=$ $\mathbb{C}$-vector sp. w/ basis χ^m $(m \in S)$

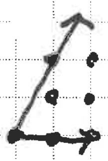
& multi: $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$

More precisely,

$$\mathbb{C}[S] = \left\{ \sum_{m \in S} c_m \chi^m \;:\; c_m \in \mathbb{C},\; c_m = 0 \text{ for all but fin. many } m\text{'s} \right\}$$

Ex. $\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1 \cdots x_n]$

$\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1} \cdots x_n^{\pm 1}]$



cone σ ⇝ $\mathbb{C}[\sigma] = \mathbb{C}[x, xy, xy^2] \cong \mathbb{C}[x, y, z] / \langle xz - y^2 \rangle$

Prop $S \subseteq M$: aff s.gp. Then

(a) $\mathbb{C}[S]$: integral domain & f.g. $\mathbb{C}$-alg

(b) Spec $\mathbb{C}[S]$: affine toric var with $\mathcal{A} \subseteq M = \mathbb{Z}S$ s.t. $S = \mathbb{N}\mathcal{A}$ ↳ $Y_{\mathcal{A}}$

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Proof. Need to show that $\mathbb{C}[Y_{\mathcal{A}}] \simeq \mathbb{C}[S]$, where

- $\mathcal{A}$: gen. of $S = \{m_1, \dots, m_s\}$
- $M = \mathbb{Z}\mathcal{A}$

$* \ \mathbb{C}[S] = \langle \chi^m : m \in S \rangle$

Define $\mathbb{C}[x_1, \cdots, x_s] \xrightarrow{\pi} \mathbb{C}[S]$ surj. w/ $\ker\pi = I_L$

$x_i \mapsto \chi^{m_i}$ $\quad (x^{l_+} - x^{l_-} \mapsto 0)$ ▨

Thm. TFAE (a) V : aff. toric var.

(b) $V = Y_{\mathcal{A}}$

(c) $V = V(I_L)$

(d) $V = \operatorname{Spec} \mathbb{C}[S]$

For (a) $\Rightarrow$ (d), $T_N \hookrightarrow V$ implies $\mathbb{C}[V] \hookrightarrow \mathbb{C}[M]$

$\therefore \mathbb{C}[V]$: subalg of $\mathbb{C}[M]$ ($\because$ If $f|_V = g|_V$, then $f = g$)

- $\mathbb{C}[V] \circlearrowleft T_N \subset \mathbb{C}[M] \circlearrowleft T_N$

$\hookrightarrow$ decomposed into $\bigoplus_{\chi^m \in \mathbb{C}[V]} \mathbb{C} \cdot \chi^m$ $\quad \left(\overset{\text{Hw}}{\simeq} \mathbb{C}[V]\right)$

Set $S = \{m \in M : \chi^m \in \mathbb{C}[V]\}$

2. Cones & normal aff. toric var.

- Fix N and $M = \mathrm{Hom}(N, \mathbb{Z})$. $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$
- $\sigma \in N_{\mathbb{R}} \simeq \mathbb{R}^n$ convex rational polyhedral cone if

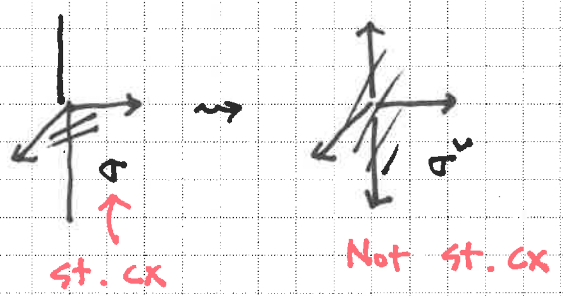
$$\sigma = \mathrm{Cone}(S) := \left\{ \sum_{u \in S} \lambda_u u \ : \ \lambda_u \geq 0 \right\} \subseteq N_{\mathbb{R}}$$

for some finite set $S \subseteq N$

lemma $\sigma^{\vee\vee} = \sigma$

- $\sigma^{\vee} := \{ m \in M_{\mathbb{R}} \ : \ \langle m, u \rangle \geq 0 \text{ for } \forall u \in \sigma \}$

σ is strongly convex if $\dim \sigma^{\vee} = n$



lemma. st. cx $\Leftrightarrow$ no positive subsp in σ.

Defn + prop + Notation.

(1) for $\overset{*0}{m} \in M$, $H_m^{+} := \{ u \in N_{\mathbb{R}} \ : \ \langle u, m \rangle \overset{>}{=} 0 \}$

(2) $\sigma \subseteq H_m^{+} \Leftrightarrow m \in \sigma^{\vee} \setminus \{0\}$

(note. $\sigma = H_{m_1}^{+} \cap \cdots \cap H_{m_s}^{+}$, $\{m_1, \cdots, m_s\}$ gen. of $\sigma^{\vee}$)

(3) face of σ := $\tau = H_m \cap \sigma$ for some $m \in \sigma^{\vee}$

(4) ray " := 1-dim. face of σ

(5) For a ray ρ of σ, $u_\rho \in N$ generator of $\rho \cap N$

(6) σ is smooth if $\{u_\rho\}$: basis of N

simplicial if " lin. indep (over $\mathbb{R}$)

Lemma (1) Every face is conv. rat. poly. cone (st)

(2) $\cap$ face $=$ face

(3) face of face $=$ face

(4) $\sigma \overset{1:1}{\underset{\text{faces}}{\longleftrightarrow}} \sigma^\vee$

$$\tau \longleftrightarrow \tau^* := \{ m \in \sigma^\vee : \langle m, u \rangle = 0 \ \forall u \in \tau \}$$

dimension reversing $(\dim \tau + \dim \tau^* = n)$

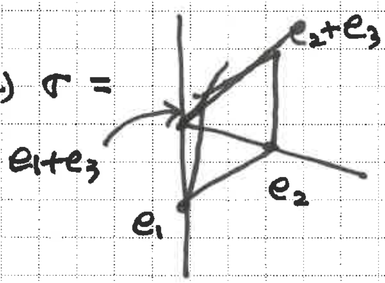
(5) (Gordan's lemma) $S_\sigma := \sigma^\vee \cap M$ is aff. s. gp.

Cor. $\operatorname{Spec} \mathbb{C}[S_\sigma] =: U_\sigma$ is aff. toric var. with $M \supset \mathcal{A}$: gen of $\sigma^\vee$

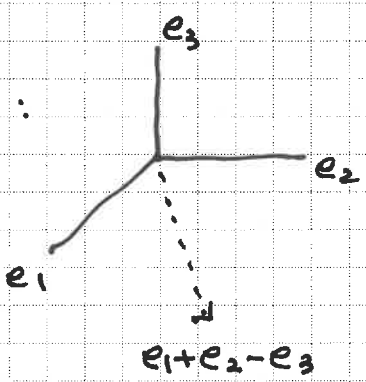
Ex. (1) $\sigma = \operatorname{cone}(e_1 \cdots e_r) \subseteq \mathbb{R}^n \rightsquigarrow \sigma^\vee = \operatorname{cone}(e_1 \cdots e_r, \pm e_{r+1} \cdots \pm e_n)$

(note. all sm. aff. toric var. are those) $\rightsquigarrow U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma] \simeq \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$

(2) $\sigma =$



$\rightsquigarrow \sigma^\vee$:



$$\therefore \mathbb{C}[S_\sigma] = \mathbb{C}[X, Y, Z, XYZ^{-1}]$$

$$\simeq \mathbb{C}[X, Y, Z, W] / \langle XY - ZW \rangle$$

same dim. ($\mathbb{Z}S \subseteq M$)

* Normality

Set $S \subseteq M$

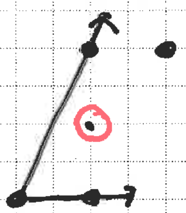
Defn if $k>0$, $m \in M$, $km \in S$ $\Rightarrow m \in S$

Thm (TFAE) V : aff. toric var

(1) V : normal

(2) $V \simeq \operatorname{Spec} \mathbb{C}[S]$, S is $\boxed{\text{saturated}}$

(3) $V \simeq U_\sigma$, $\sigma \subseteq N_{\mathbb{R}}$: st rat conv. poly. cone



Not saturated

Idea : Assume V : normal & $k>0$ $m\in M$ $km \in S$.

$\rightsquigarrow Y^k - \chi^{km} \in \mathbb{C}[S][Y]$ has a root χ^m in $\mathbb{C}(S) = \mathbb{C}(M)$

$\Rightarrow \chi^m \in \mathbb{C}[S]$ by the normality

$\boxed{3}$ Toric morphisms

Let $V_i = \operatorname{Spec} \mathbb{C}[S_i]$ $(i=1,2)$. Recall : $\phi : V_1 \to V_2$

$\leftrightarrow \phi^* : \mathbb{C}[S_2] \to \mathbb{C}[S_1]$

Defn We call ϕ toric if ϕ^* is induced by $S_2 \to S_1$

Prop (a) ϕ is toric $\Leftrightarrow$ $\phi(T_{N_1}) \subseteq T_{N_2}$ and $\phi|_{T_{N_1}}$: gp. homo.

(b) Toric morphism is equivariant

$(\phi(t\cdot x) = \phi(t)\cdot\phi(x))$

↳ (Hint)

$$\begin{array}{ccc} \mathbb{C}[S_2] & \xrightarrow{\phi^*} & \mathbb{C}[S_1] \\ \downarrow & \exists\, \tilde{\phi}^* & \downarrow \\ \mathbb{C}[M_2] & \longrightarrow & \mathbb{C}[M_1] \end{array}$$

$\operatorname{Spec} \mathbb{C}[M] \simeq T_N$

$\{ \mathbb{C}[M] \xrightarrow{\phi_t} \mathbb{C} \} \sim \{ t \}$

$\chi^m \mapsto \chi^m(t)$

$\leftrightarrow \{ M \to \mathbb{C}^* \}$

Prop Let $\sigma_i \subseteq (N_i)_{\mathbb{R}}$: st cx rat poly cone

$\bar{\phi} : N_1 \to N_2$

(1) $\bar{\phi}$ induces homo $T_{N_1} \xrightarrow{\phi} T_{N_2}$ via $T_N = \mathrm{Hom}(M, \mathbb{C}^*) \simeq N \otimes_{\mathbb{Z}} \mathbb{C}^*$

$$\begin{tikzcd} T_{N_1} \arrow[r, "\phi"] \arrow[d, phantom, "\cap" rotate=-90] & T_{N_2} \arrow[d, phantom, "\cap" rotate=-90] \\ U_{\sigma_1} \arrow[r, dotted, "?"] & U_{\sigma_2} \end{tikzcd}$$

(2) ϕ extends to a toric morphism $U_{\sigma_1} \to U_{\sigma_2}$

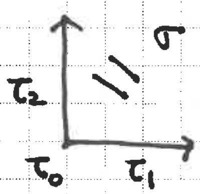
$\Leftrightarrow \bar{\phi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$

$\boxed{4}$ Faces : let $\tau < \sigma \subseteq N_{\mathbb{R}}$ $\tau = H_m \cap \sigma^\vee$ for $m \in \sigma^\vee$

Prop $\mathbb{C}[\tau^\vee \cap M] = \mathbb{C}[\sigma^\vee \cap M]_{\chi^m}$

$\therefore U_\tau = (U_\sigma)_{\chi^m} \rightsquigarrow$ $N \xrightarrow[id]{} N$, $\tau \hookrightarrow \sigma$ induces an embedding $U_\tau \hookrightarrow U_\sigma$

Ex.

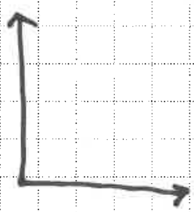


$U_{\tau_0} \simeq \mathbb{C}^* \times \mathbb{C}^*$

$U_{\tau_1} \simeq \mathbb{C} \times \mathbb{C}^*$

$U_{\tau_2} \simeq \mathbb{C}^* \times \mathbb{C}$

$U_\sigma \simeq \mathbb{C} \times \mathbb{C}$



$\mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$

$\boxed{5}$ Projective varieties : Introduction

- $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*$
  $\cup\!\mid$
  V : projective variety if $V = V(I)$, I : homogeneous ideal of $\underbrace{\mathbb{C}[x_0, \cdots, x_n]}_{S}$

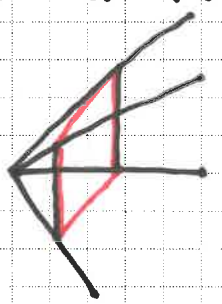
- $\mathbb{C}[V] := S / I(V)$ homogeneous coord. ring

- $\mathbb{C}[V]_d := S_d / I(V)_d \rightsquigarrow \mathbb{C}[V] = \bigoplus_d \mathbb{C}[V]_d$

- $\hat{V}$: affine cone of V

Ex Consider $\hat{V}(xy - zw) \subseteq \mathbb{C}^4$.

$\hat{V}$



singular

$\longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \simeq V$

* There is no "regular ftn" on proj. var. (by Liouville)

For $f, g \in S_d$, $\frac{f}{g} : \mathbb{P}^n \setminus V(g) \to \mathbb{C}$ is well-def.

$\mathbb{P}^n \dashrightarrow \mathbb{C}$ (called rational ftn on $\mathbb{P}^n$)

Defn. $\mathbb{C}(V) := \left\{ \frac{f}{g} \in \mathbb{C}(\hat{V}) : f, g \in \mathbb{C}[\hat{V}]_d \text{ for some } d \right\}$

called "rational ftn field of V"