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Lecture III (12/11 ~ 12/12)

- Review : normal affine toric varieties
- projective toric varieties
  - Proj construction
  - Affine cone
  - Affine pieces
- Lattice polytopes
  - smooth, normal, very ample
  - affine pieces
  - gluing : normal fan
- Fans and normal toric varieties
  - Examples
  - Orbit - cone correspondence
  - Orbit closure
  - Equivariant maps
  - Blow-up

1 Review

- Three constructions of aff. toric var's :

I. finite lattice pts : For $\mathcal{A} = \{m_1, \cdots, m_s\} \subseteq M = \mathbb{Z}^n$ $(= \mathrm{Hom}(T_N, \mathbb{C}^*))$

$$Y_{\mathcal{A}} := \text{Zariski closure of } \Phi_{\mathcal{A}}(T_N)$$

where $\Phi_{\mathcal{A}} : T_N = (\mathbb{C}^*)^n \to \mathbb{C}^s$

$$t = (t_1, \cdots, t_n) \mapsto (t^{m_1}, \cdots, t^{m_s})$$

* Same for $\mathbb{P}^{s-1}$ $(\because \mathrm{Im}\, \Phi_{\mathcal{A}} \subseteq \mathbb{C}^s \setminus \{0\})$

II. Toric ideal (: prime ideal generated by binomials) (of $\mathbb{C}[x_1, \cdots, x_s]$)

$$0 \to L = \ker \psi \to \mathbb{Z}^s \to M \to 0$$

$l = (l_1, \cdots, l_s) \in L$ s.t $\sum l_i m_i = 0$; $e_i \mapsto m_i$

Setting $l = l_+ - l_-$, $l_+, l_- \in \mathbb{N}^s$, it is easy to show that $x^{l_+} - x^{l_-}$ vanishes on $Y_{\mathcal{A}}$

No common divisor

$(\because$ For $(t^{m_1}, \cdots, t^{m_s}) \rightsquigarrow t^{\sum m_i l_i} = 1)$

III. Affine semigroup (fin. gen. abelian semigp.)

$\rightsquigarrow$ looks like $\mathbb{N} \cdot \mathcal{A}$

For $S := \mathbb{N} \cdot \mathcal{A}$, define $\mathbb{C}[S] := \left\{ \sum_{m \in S} c_m \chi^m : \begin{array}{l} c_m \in \mathbb{C} \\ c_m = 0 \text{ except for} \\ \text{finite } m\text{'s} \end{array} \right\}$

w/ $\chi^m \cdot \chi^{m'} := \chi^{m+m'}$

Then $\mathrm{Spec}\, \mathbb{C}[S] = Y_{\mathcal{A}}$

$(\because \mathbb{C}[x_1, \cdots, x_s] \xrightarrow{\pi} \mathbb{C}[S]$ s.t. $x_i \mapsto \chi^{m_i} \rightsquigarrow \ker \pi = I_L)$

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• Cones & normal aff. toric var

Thm. $V = \mathrm{Spec}\, \mathbb{C}[S]$ is normal iff S is saturated*

(* : For $S \subseteq M$, if $k>0$ and $km \in S$, $m \in M$, then $m \in S$)

Thm Any saturated aff. semi g.p $S = \sigma^\vee \cap M$, σ : st. conv. rat. poly. cone
(Gordan's lemma)

* $\sigma = \mathrm{cone}(S) := \left\{ \sum_{u \in S} \lambda_u \cdot u \ : \lambda_u \geq 0 \right\} \subseteq N_{\mathbb{R}}$

$\sigma^\vee := \{ m \in M_{\mathbb{R}} : \langle m, u \rangle \geq 0 \ \forall u \in \sigma \}$ dual cone.

Thm (Sumihiro) $\forall$ normal aff. toric var $= U_\sigma$ where

$U_\sigma := \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M]$

• σ is smooth iff U_σ smooth.
simplicial iff U_σ : orbifold

• $\tau < \sigma$ f $\tau = H_m \cap \sigma$ for $m \in \sigma^\vee$.

$\sigma \longleftrightarrow \sigma^\vee$

$\tau \longleftrightarrow \tau^* := \{ m \in \sigma^\vee : \langle m, u \rangle = 0 \ \forall u \in \tau \}$

order reversing
dimension

• Toric morphism between $V_i = \mathrm{Spec}\, [S_i]$

* $\phi : V_1 \to V_2$ is toric iff $\phi(T_{N_1}) \subseteq T_{N_2}$ & $\phi|_{T_{N_1}}$: homo.

$\rightsquigarrow$ ϕ is equivariant.

$$\begin{array}{ccc} \mathbb{C}[S_2] & \xrightarrow{\phi^*} & \mathbb{C}[S_1] \\ \downarrow & & \downarrow \\ \mathbb{C}[M_2] & \xrightarrow{\exists \check{\phi}^*} & \mathbb{C}[M_1] \end{array}$$

If $\sigma_i \subseteq (N_i)_{\mathbb{R}}$: st. rat. cx. poly.

Let $\bar{\phi} : N_1 \to N_2$. $\rightsquigarrow$

$$\begin{array}{ccc} T_{N_1} & \xrightarrow{\phi} & T_{N_2} \\ \cap & & \cap \\ U_{\sigma_1} & \dashrightarrow & U_{\sigma_2} \end{array}$$

ϕ extend to this iff $\bar{\phi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$

Ex. Faces $\tau \prec \sigma \subseteq N_{\mathbb{R}}$. $\phi : N \xrightarrow{id} N$

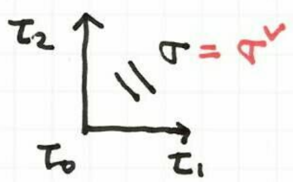
$(\tau = H_m \cap \sigma)$ $\qquad \sigma_1 = \tau \quad \sigma_2 = \sigma$

$U_\tau \to U_\sigma$ (Note. $\tau^\vee \cap M = \langle \sigma^\vee \cap M, -m \rangle$

$\pm m$ (Inverting m).

$\rightsquigarrow \underbrace{\mathbb{C}[\tau^\vee \cap M]}_{} = \mathbb{C}[\sigma^\vee \cap M]_{\chi^m}$

nonzero pts of χ^m in U_σ.



$U_\sigma = \mathbb{C}^2$

$U_{\tau_1} = \{(x, y) \in U_\sigma : y \neq 0\} = \mathbb{C} \times \mathbb{C}^*$

$U_{\tau_2} = \mathbb{C}^* \times \mathbb{C}$

$U_{\tau_0} = \mathbb{C}^* \times \mathbb{C}^*$

$\boxed{2}$ Projective toric varieties

• Projective = projectifying aff. I.e., given $\mathcal{A} = \{m_1, \dots, m_s\} \subseteq M$

$$T_N \xrightarrow{(\chi^{m_1}, \dots, \chi^{m_s})} (\mathbb{C}^*)^s \subseteq \mathbb{C}^s \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^{s-1}$$

$$\begin{array}{ccc} & \cup\!\!\shortmid & \cup\!\!\shortmid \\ & Y_{\mathcal{A}} & X_{\mathcal{A}} \end{array}$$

$X_{\mathcal{A}}$:= Zariski closure of image.

* Given $X \subseteq \mathbb{P}^{s-1}$, $\underbrace{\pi^{-1}(X) \cup \{0\}}_{=: \hat{X}}$ is called the affine cone over X

- $\mathbb{C}[\hat{X}] = \mathbb{C}[X]$ homog. coordinate ring

$\curvearrowleft$ Proj construction : $S := \mathbb{C}[X] = \bigoplus_{d \geq 0} S_d$

$S_+ := \bigoplus_{d > 0} S_d$.

$\operatorname{Proj} S := \{ p \subset S : \text{prime \& homog.}, S_+ \not\subseteq p \}$

* If $S_+ \subset P$, then $V(p) = \emptyset$

Note on Proj S

(ex. If $a = \langle f_1 \rangle$, then $p = \langle f_1, \dots, f_k \rangle \dots$)

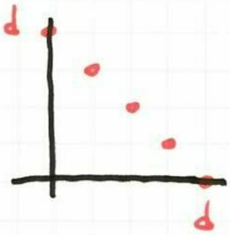
• For $\underset{\text{homog}}{a \subset S}$, set $V(a) := \{ p \in \operatorname{Proj} S : a \subseteq p \}$

$\rightsquigarrow$ Zariski topology on Proj S $\Leftrightarrow$ $V(a)$ is closed. $\forall a \subseteq S$.

(Hw: How does a closed pt look like ?)

* Projective Nullstellensatz

$$\{\text{closed subvari's of } \mathbb{P}^n\} \overset{1:1}{\longleftrightarrow} \{\text{homog. radical ideals}\}$$

Ex. (1) $\mathcal{A} =$  $\left\{ \binom{d}{0}, \binom{d-1}{1}, \cdots, \binom{1}{d-1}, \binom{0}{d} \right\}$

$$(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^{d+1} \to \mathbb{P}^d$$

「Hw: defining equation?」

$$(t,s) \mapsto [t^d, t^{d-1}s, \cdots, ts^{d-1}, s^d]$$

(2) $\mathcal{B} =$  $\{0, 1, \cdots, d\}$

$$\mathbb{C}^* \to (\mathbb{C}^*)^{d+1} \to \mathbb{P}^d$$

$$t \mapsto [1, t, \cdots, t^d]$$

* $X_{\mathcal{A}} = X_{\mathcal{B}}$ (same curve)

* $\dim Y_{\mathcal{A}} = 2$ & $Y_{\mathcal{A}} = \hat{X}_{\mathcal{A}}$, but $\dim Y_{\mathcal{B}} = 1$ & $Y_{\mathcal{B}} \neq \hat{X}_{\mathcal{B}}$.

Prop. (TFAE) (a) $Y_{\mathcal{A}}$ is aff. cone of $X_{\mathcal{A}}$

(b) $I_{Y_{\mathcal{A}}}$ is homogeneous

easy

(c) $\{m_1, \cdots, m_s\}$ is lying on some hyperplane not containing 0

($\Leftrightarrow$ $\exists\, u \in N$ and $k \in \mathbb{Z}_{>0}$ s.t. $\langle u, m_i \rangle = k$ for $\forall i$)

(b) $\leftrightarrow$ (c) $L := \ker(m_1, \cdots, m_s) \ni l = (l_1, \cdots, l_s) = l_+ - l_-$ & $\sum l_i m_i = 0$ $\rightsquigarrow x^{l_+} - x^{l_-}$

$x^{l_+} - x^{l_-}$: homog $\Leftrightarrow$ $l \cdot (1, 1, \cdots, 1) = 0$ for $\forall\, l \in L$

$\Leftrightarrow (1, 1, \cdots, 1) \in$ row space of $(m_1, \cdots, m_s)$

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For $L \to \mathbb{Z}^s \xrightarrow{A} M \to 0$, tensor $\mathbb{Q}$ and take the dual:

$e_i \mapsto m_i$

$u \mapsto \langle u, m_i \rangle$

$N_{\mathbb{Q}} \to \mathbb{Q}^s \longrightarrow \mathrm{Hom}(L_{\mathbb{Q}}, \mathbb{Q}) \to 0$

A^T : column space contains $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ i.e., $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathrm{Im} A^T$.

$\therefore \exists\, u \in N_{\mathbb{Q}}$ s.t. $\langle m_i, u \rangle = 1$, $\forall i = 1, \cdots, s$. ∎

$\boxed{3}$ Affine covers of $X_{\mathcal{A}}$: For $\mathcal{A} = \{m_1, \cdots, m_s\}$, denote by $x_1, \cdots, x_s$ the homog. coord's.

- Set $U_i := \{[x_1, \cdots, x_s] \in \mathbb{P}^{s-1} : x_i \neq 0\}$

Prop $X_{\mathcal{A}} \cap U_i = Y_{\mathcal{A}_i}$ where $\mathcal{A}_i := \mathcal{A} - m_i = \{m_j - m_i : j = 1, \cdots, s\}$

↳ $(\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^s$

$t \longmapsto [\underbrace{\chi^{m_1}(t)}_{t^{m_1}}, \cdots, \underbrace{\chi^{m_s}(t)}_{t^{m_s}}] = [t^{m_1 - m_i}, \cdots, 1, \cdots, t^{m_s - m_i}]$ ∎

Prop. $X_{\mathcal{A}} = \bigcup_{\substack{m_i : \text{vertex} \\ \text{of } \mathrm{Conv}(\mathcal{A})}} X_{\mathcal{A}} \cap U_i$

↳ Set J = index set of vertices, $\subset \{1, \cdots, s\} =: I$.

ETS $U_i \subseteq U_j$ for some $j \in J$.

$X_{\mathcal{A}} \cap U_i \cap U_j$

$\underbrace{\qquad}_{\shortparallel}$

$\mathbb{C}[S_i] = \mathbb{C}[S_i]_{m_i \cdots}$ $\chi^{m \cdots}$

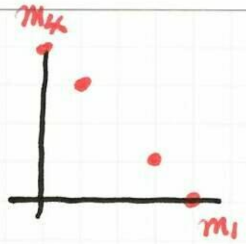
- $m_i = \sum_j q_j m_j$, $q_j \in \mathbb{Q}$ & $\sum q_j = 1$

$\rightsquigarrow k m_i = \sum_j k_j m_j$, $k = \sum k_j$

$\rightsquigarrow \sum_j k_j (m_j - m_i) = 0$ $\rightsquigarrow$ For $k_j \neq 0$, $k_j(m_i - m_j) \in \mathcal{A}_i$

$\rightsquigarrow m_i - m_j \in \mathcal{A}_i \rightsquigarrow \chi^{m_j - m_i}$: invertible

Example



$$A = \mathrm{Col}\begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix} \qquad X_A \subseteq \mathbb{P}^3$$

$$(\mathbb{C}^*)^2 \longrightarrow (\mathbb{C}^*)^4$$
$$(s,t) \longmapsto [s^4, s^3t, st^3, t^4]$$

- $X_A \cap U_1 = \overline{\left\{ [1, \frac{t}{s}, \frac{t^3}{s^3}, \frac{t^4}{s^4}] \right\}} \subseteq \mathbb{C}^3$

  $\simeq \mathbb{C} \quad (\because \mathbb{C}[S_1] \simeq \mathbb{C}[\frac{t}{s}])$

- Similarly $X_A \cap U_4 \simeq \mathbb{C}$.

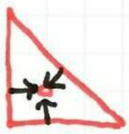
* Note : X_A is normal ($\because$ smooth) but not projectively normal (why?) ·· Hw

$\boxed{4}$ Lattice polytopes

Given $P = \mathrm{Conv}(A) \subseteq M_{\mathbb{R}}$, we have a facet presentation

$$P = \{ m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F, \ F : \text{facet} \}$$

inward facet normal



$$\left.\begin{aligned} y &\geq -1 \\ x &\geq -1 \\ -x-y &\geq -1 \end{aligned}\right\}$$

Defn P is **simplex** if P has $\dim P + 1$ vertices

dual: **simplicial** if $\forall$ proper face is a simplex

simple if $\forall$ vertex is contained in exactly $\dim P$ facets

We want to define $X_P := X_{P\cap M}$ could be too "small"

- $\underbrace{P\cap M + \cdots + P\cap M}_{k} \subseteq kP\cap M \quad k\geq 1$

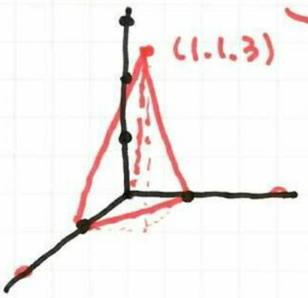
Defn. P is normal if $k(P\cap M) = kP\cap M$

very ample if for every vertex v of P,

$S_{P,v} := \mathbb{N}(P\cap M - v) \subseteq M$ is saturated

Ex. (non-normal)

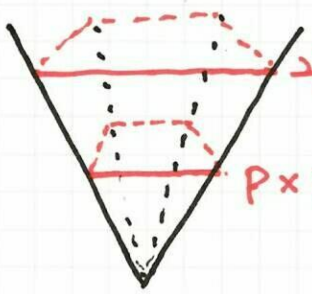
$P = \mathrm{Conv}(\underbrace{0, e_1, e_2, e_1+e_2+3e_3}_{= P\cap M})$



For $2P\cap M = \mathrm{Conv}(0, 2e_1, 2e_2, 2e_1+2e_2+6e_3)$,

$\underset{\not\in\ 2(P\cap M)}{\overset{\in}{e_1+e_2+e_3}} = \frac{1}{6}\cdot 0 + \frac{1}{3}(2e_1) + \frac{1}{3}(2e_2) + \frac{1}{6}(2e_1+2e_2+6e_3)$

* Let $C(P) = \mathrm{Cone}(P\times\{1\}) \subseteq M_{\mathbb{R}}\times\mathbb{R}$



P is normal iff $C(P)\cap M\times\mathbb{Z}$ is generated by elts of ht 1.

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Theorem. (1) kP is normal for $\forall k \geq \dim P - 1$

(2) Normal $\Rightarrow$ very ample.

$\hookrightarrow$ (2) Fix $m_0 \in P$: vertex.

Fix $m \in M$ s.t. $km \in S_{P,m_0} := \mathbb{N}(P \cap M - m_0)$

$\rightarrow$ Need to prove $m \in S_{P,m_0}$

Let $km = \sum_{m' \in P \cap M} a_{m'} \cdot (m' - m_0)$ and let $d \in \mathbb{N}$ s.t. $kd \geq \sum a_{m'}$

$\Rightarrow km + kdm_0 = \sum a_{m'} \cdot m' + (kd - \sum a_{m'}) m_0 \in kdP$

($\because$ sum of coef $= kd$)

$\Rightarrow m + dm_0 \in dP$

$= \sum_1^d m_i$

normality used

$\therefore m = \sum_1^d (m_i - m_0) \in S_{P,m_0}$

Easy prop. Smooth $\Rightarrow$ Very ample.

Conjecture (Oda) Smooth $\overset{?}{\Rightarrow}$ normal ($\Leftrightarrow$ sm. proj $\overset{?}{\Rightarrow}$ proj. normal)

$\boxed{5}$ Toric variety of P

* Assume that P is very ample and $P \cap M = \{m_1, \cdots, m_s\}$

Thm For m_i : vertex, set $\sigma_i = \mathrm{Cone}(P \cap M - m_i)^\vee \subseteq N_{\mathbb{R}}$

Then $X_{P \cap M} \cap U_i \simeq U_{\sigma_i}$

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Q : How affine pieces fit together? $\rightsquigarrow$ normal "fan"

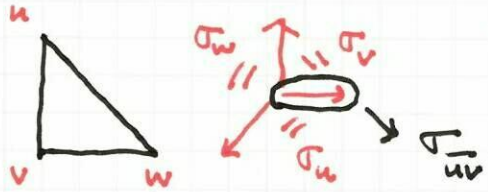
↳ for $\underset{\text{face}}{Q \prec P}$ and $\sigma_Q := \text{cone}(u_F : Q \prec F) \subseteq N_{\mathbb{R}}$

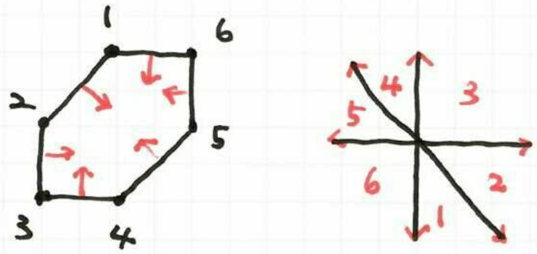
Thm $\Sigma_P := \{ \sigma_F : F \text{ face of } P \}$ is a fan

collection of cones s.t.

- $\sigma \cap \tau \in \Sigma$
- $\tau \prec \sigma \Rightarrow \tau \in \Sigma$

Example





Prop. Let $v \neq w$ vertices of P. Denote by $U_v, U_w \subseteq \mathbb{P}^{s-1}$ coord. charts for v, w so that

$$\begin{cases} X_{P\cap M} \cap U_v = U_{\sigma_v} = \operatorname{Spec}\mathbb{C}[\sigma_v^\vee \cap M] \\ X_{P\cap M} \cap U_w = U_{\sigma_w} \end{cases}$$

$\sigma_v \cap \sigma_w = \sigma_Q$

Let Q : smallest face containing v, w

Then $X_{P\cap M} \cap U_v \cap U_w = U_{\sigma_Q}$

↳ Note $\chi^{w-v} \in \mathbb{C}[\sigma_v^\vee \cap M]$ $\quad U_{\sigma_v}$

$\chi^{v-w} \in \mathbb{C}[\sigma_w^\vee \cap M]$ $\overset{\rightsquigarrow}{\text{spec}}$ U_{σ_w}

$\underbrace{X_{P\cap M} \cap U_v \cap U_w}_{U_{\sigma_v}} = \operatorname{Spec}\mathbb{C}[\sigma_v^\vee \cap M]_{\chi^{w-v}}$

Easy : $\sigma_Q = H_{w-v} \cap \sigma_v$ $\quad$ ($\Leftrightarrow$ semigroup contains $\mathbb{Z}\langle v-w\rangle$)

Conclusion When P : very ample, we build $X_{P \cap M}$ from $\{U_{\sigma_v}\}$ in a way completely determined by Σ_P.

* Observation P and kP give the same fan, and hence same abstract toric variety.

6 Properties of X_P

Thm (a) X_P is normal

(b) X_P is proj. normal iff P is normal

Thm (TFAE) (a) X_P smooth

(b) Σ_P smooth

(c) P smooth

Lecture IV (12/18-19)

[1] Fans and normal toric varieties

General Defn of toric varieties : (abstract, separated) variety X containing T_N as Zariski open subsets s.t $T_N \circlearrowright$ extends to X.

∃ affine cover ↝ Hausdorff in classical top.

Defn. A fan Σ in $N_{\mathbb{R}}$ is a finite set of st. cx. rat. poly. cones s.t.

- $\sigma \in \Sigma$ and $\tau < \sigma \Rightarrow \tau \in \Sigma$
- $\sigma, \tau \in \Sigma \Rightarrow \sigma \cap \tau < \sigma, \tau$

* Given $\Sigma \ni \sigma$, get U_σ. Glue U_σ and U_τ along $U_{\sigma \cap \tau}$.
This gives an abstract variety X_Σ.

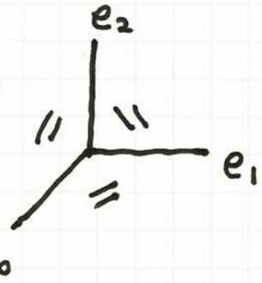
open in U_σ & U_τ

Thm (a) X_Σ is normal toric variety

(b) If $P \subseteq M_{\mathbb{R}}$: full dim'l lattice polytope, then $X_{\Sigma_P} = X_P$

(c) (Sumihiro) Every normal toric var. comes from a fan

Ex



$N = \mathbb{Z}^n$

$e_0 = -\sum_{i=1}^{n} e_i$

$\Sigma = \{$ cones gen. by proper subsets of $\{e_0, \cdots, e_n\} \}$

$\rightsquigarrow X_\Sigma = \mathbb{P}^n \quad (\because \Sigma = \Sigma_{\Delta_n})$

Ex $X_{\Sigma_1 \times \Sigma_2} \simeq X_{\Sigma_1} \times X_{\Sigma_2}$

Ex Let $q_0, \cdots, q_n > 0$ in $\mathbb{Z}$ with $\gcd = 1$

- $N := \mathbb{Z}^{n+1} / \mathbb{Z}\cdot(q_0, \cdots, q_n) \cong \mathbb{Z}^n$

- u_i = image of e_i $\rightsquigarrow$ $\sum q_i u_i = 0$

- $\Sigma := \{$ cones gen. by proper subsets of $\{u_0, \cdots, u_n\}\}$

$\rightsquigarrow X_\Sigma = \mathbb{P}(q_0, \cdots, q_n)$.

Defn (a) Σ : smooth if $\forall \sigma$: smooth

(b) Σ : simplicial if $\forall \sigma$: simplicial

(c) $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$ support of Σ

(d) Σ : complete if $|\Sigma| = N_{\mathbb{R}}$

Thm (a) X_Σ smooth iff Σ smooth

(b) X_Σ cpt (w.r.t classical top) iff Σ : complete

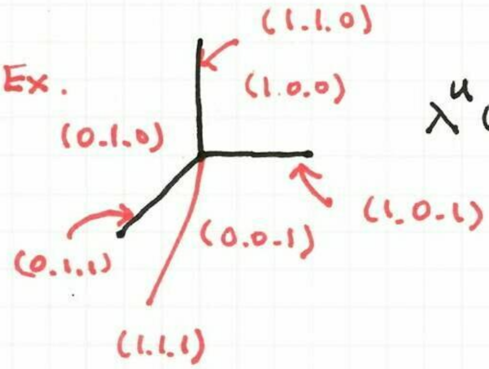
2 Orbit - cone correspondence

· Recall N : lattice of one-parameter subgroup

$$\rightsquigarrow u \in N \ : \ \lambda^u : (\mathbb{C}^*) \hookrightarrow (\mathbb{C}^*)^n$$
$$t \mapsto (t^{u_1}, \cdots, t^{u_n})$$

Compute $\lim_{t \to 0} \lambda^u(t)$.

Ex.

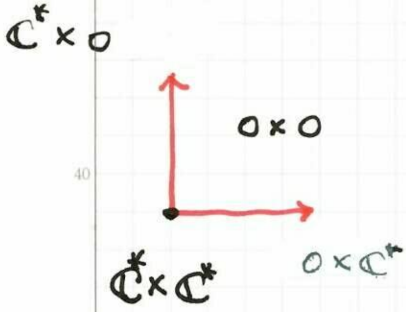


$\lambda^u(t) \mapsto [1, t^a, t^b] \in \mathbb{P}^2$

Prop. $\sigma \subseteq N_{\mathbb{R}}$ st. cx. rat. poly. cone

· orbits $\leftrightarrow$ cones
size-reversing

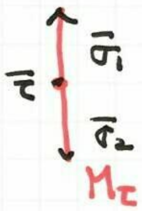
- $u \in \sigma$ iff $\lim_{t \to 0} \lambda^u(t)$ exists in U_σ
- If $u \in \mathring{\sigma}$, then the limit is the identity of $O(\sigma)$



- $O(\tau)$: torus
- $U_\sigma = \bigcup_{\tau < \sigma} O(\tau)$ ← $O(\sigma)$가 제일 작다.
- $V(\tau) := \overline{O(\tau)} = \bigcup_{\tau < \sigma} O(\sigma) = O(\tau) \cup$ smaller stuff

How to describe $V(\tau)$? Ex.

$V(\tau) \subseteq \mathbb{P}^2$?



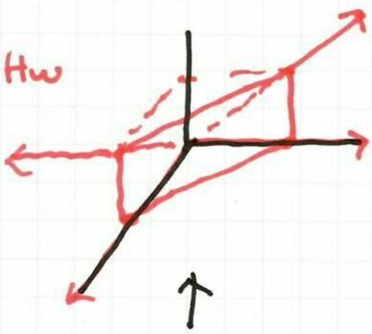
- Character lattice of $O(\tau)$ is $\tau^\perp \cap M \hookrightarrow M$
- (dual) $N \to N(\tau) := N / \operatorname{span}(\tau) \cap N$

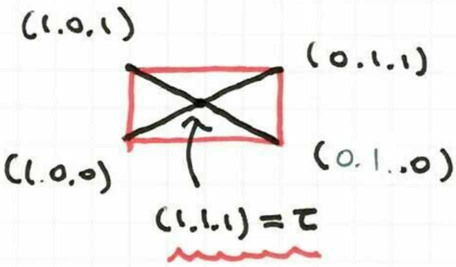
$\sigma \mapsto \bar{\sigma}$

Set $\mathrm{Star}(\tau) := \{ \bar{\sigma} : \tau < \sigma \}$

Thm $\mathrm{Star}(\tau)$ is a fan in $N(\tau)_{\mathbb{R}}$ & $X_{\mathrm{Star}(\tau)} = V(\tau)$

Hw





fan consisting of four max. dim'l cones.

What is $V(\tau) \subseteq X_\Sigma$?

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3 Introduction to Divisors

Defn X : irred. normal variety

(1) Prime divisor $D \subseteq X$ is a codim 1 irred. subvar. $\subseteq X$

(2) $\mathrm{Div}(X)$ is a free abelian group generated by all prime divisors. An elt $D = \sum a_i D_i$ in $\mathrm{Div}(X)$ is called Weil divisor

$\mathrm{Supp}(D) := \bigcup_{a_i \neq 0} D_i$

Consider $\mathbb{C}(X) := \left\{ \begin{array}{l} \text{regular fns defined on some open subsets} \\ \text{of } X \end{array} \right\}$

function field

- If $X = \mathrm{Spec}\, R$, then $\mathbb{C}(X)$: field of fractions of R
- If $U \subseteq X$ open, $\mathbb{C}(X) = \mathbb{C}(U)$ ⇝ birational invariant

For each prime divisor $D \subseteq X$, one can define

$$\nu_D : \mathbb{C}(X)^* \to \mathbb{Z} \quad \text{discrete valuation}$$

- homo.
- $\nu(x+y) \geq \min(\nu(x), \nu(y))$

($\cdot\ \nu(0) := \infty$)

s.t. $\nu_D(f) :=$ order of vanishing of f on D

Ex. $D \subseteq \mathbb{C}^n$ given by $D = V(g)$.

For any $f = g^k \cdot \frac{h_2}{h_1}$, h_i is not divisible by g,

$\nu_D(f) = k$ ($k > 0$ ⇝ $f \equiv 0$ on D

$k < 0$ ⇝ f is not defined on D ⇝ pole of order k

$k = 0$ ⇝ nowhere vanishing on an open subset of D)

Note. Formally, ν_D is defined as follows.

$$\mathcal{O}_{X,D} := \{ \phi \in \mathbb{C}(X) : \phi \text{ is defined on some open } U \subseteq X \ \& \ U \cap D \neq \emptyset \}$$

$U \cap D$ open in D (hence dense)

Ex. $X = \operatorname{Spec} R$
$D = V(p)$ codim 1

$\Rightarrow \mathcal{O}_{X,D} = R_p$

Prop. $\mathcal{O}_{X,D}$: DVR ($\nu : K^* \to \mathbb{Z}$ DV.
$\Rightarrow \{ x \in K^* : \nu(x) \geq 0 \} =: R$
called DVR)

- $\mathcal{O}_{X,D} = \mathcal{O}_{U, U\cap D}$ ($\because$ reduce to affine cases)

$\therefore \mathcal{O}_{X,D} \simeq R_p$ N.T.S. it is DVR

- pR_p : maximal in R_p. Then (R_p, m_p) : Noetherian local domain of dim 1

$pR_p = m_p$

(Thm R_p : DVR $\Leftrightarrow$ normal $\Leftrightarrow$ m_p principal)

$m_p = \langle f \rangle$.

Noetherian
Need assumption

Prop. For $f \in \mathbb{C}(X)^*$, $\nu_D(f) = 0$ for all but fin. many D's.

↳ Let $V \subseteq X$ s.t f is defined on V. & $U := f|_V^{-1}(\mathbb{C}^*)$

$X \setminus U = D_1 \cup \cdots \cup D_s \cup$ higher codim cpnts

codim 1

$\rightsquigarrow \nu_D(f) \neq 0$ only if $D = D_i$ for some i.

Defn For $f \in \mathbb{C}(X)^*$, $\operatorname{div}(f) := \sum_D \nu_D(f) \cdot D$ is called a principal divisor

- $\operatorname{Div}_0(X) := \{ \operatorname{div}(f) : f \in \mathbb{C}(X)^* \} \underset{\text{subgp}}{\subseteq} \operatorname{Div}(X)$

- $\operatorname{Div}(X) / \operatorname{Div}_0(X) =: C\ell(X)$ class group

- $D \sim E$ linearly equiv. if $[D] = [E] \in C\ell(X)$

- $D \in \operatorname{Div}(X)$ is Cartier if $\exists$ open cover $\{U_i\}$ of X s.t.

  $D|_{U_i} = \operatorname{div}(f_i)|_{U_i}$ for some $f_i \in \mathbb{C}(X)^*$. We call $\{(U_i, f_i)\}$

  local data for D.

* Note $D = \sum_{D_i} a_i D_i$

$\rightsquigarrow D|_U := \sum_{D_i \cap U \neq \phi} a_i (D_i \cap U)$

$D \geq 0 \Leftrightarrow a_i \geq 0$ (effective)

Note. (1) $\operatorname{div}(f) \geq 0 \Leftrightarrow f \in \mathcal{O}_X(X)$

(2) $\operatorname{div}(f) = 0 \Leftrightarrow f \in \mathcal{O}_X^*(X)$

$\boxed{4}$ Computing Cl :

Thm R : UFD and $X = \mathrm{Spec}(R)$. Then $Cl(X) = 0$

↳ If D prime in $\mathrm{Div}(X)$, $D = \mathbb{V}(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of codim 1 $\rightsquigarrow \mathfrak{p} = \langle f \rangle \rightsquigarrow \mathrm{div}(f) = D$.

- **Ex** : $Cl(\mathbb{C}^n) = 0$.

Thm X : normal U : open in X, $D_1 \cdots D_s$: irred cpnt of $X \setminus U$

Then

$$\bigoplus_{j=1}^{s} \mathbb{Z} D_j \to Cl(X) \to Cl(U) \to 0$$

exact

↳ surj : $D' = \sum a_i D_i' \in \mathrm{Div}(U)$

$\rightsquigarrow D := \sum a_i \overline{D_i'} \in \mathrm{Div}(X)$ s.t $D|_U = D'$

exact : If $D \in \mathrm{Div}(X)$ s.t $D|_U = \mathrm{div}(f)|_U$,

$\rightsquigarrow (D - \mathrm{div}(f))|_U = 0 \rightsquigarrow \underbrace{D - \mathrm{div}(f)}_{\sim D} \in \bigoplus \mathbb{Z} D_j$.

Ex. $\mathbb{P}^1 = \underset{U}{\mathbb{C}} \cup \underset{D_1}{\{\infty\}}$

$$\mathbb{Z}\langle\infty\rangle \xrightarrow[\therefore\ \text{surj}]{} Cl(\mathbb{P}^1) \to \cancel{Cl(\mathbb{C})}\ 0 \to 0$$

- If $a\langle\infty\rangle \mapsto 0$, then $a\langle\infty\rangle = \mathrm{div}(f)$ for som $f \in \mathbb{C}(\mathbb{P}^1)$

$\rightsquigarrow \mathrm{div}(f)|_{\mathbb{C}} = 0 \rightsquigarrow f$: nowhere vanishing hol. on $\mathbb{C}$

$\rightsquigarrow$ const.

$\therefore a = 0$

Thm. X : smooth

$\Rightarrow \mathrm{Pic}(X) = Cl(X)$

Lecture V

1 Computing $Cl(X_\Sigma)$ and $Pic(X_\Sigma)$

Recall

$$M \xrightarrow{\alpha} \bigoplus_\rho \mathbb{Z}D_\rho \to Cl(X_\Sigma) \to \cancel{Cl(T_N)} \to 0 \quad \text{exact}$$

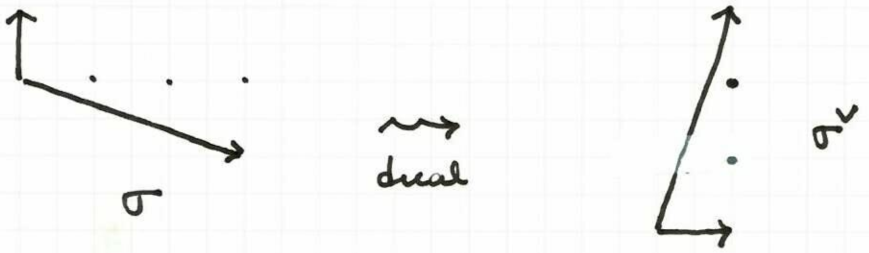
$$m \mapsto \mathrm{div}(\chi^m) = \sum_\rho \langle m, u_\rho \rangle D_\rho$$

when $\{u_\rho\}$ spans $N_\mathbb{R}$

Note α is dual map of

$$\mathbb{Z}D_\rho \mapsto N$$
$$D_\rho \mapsto u_\rho$$

Example $\sigma = \mathrm{cone}(de_1 - e_2, e_2)$



$\Rightarrow \mathbb{Z}D_\rho \to N = \mathbb{Z}^2$

$$\begin{pmatrix} d & 0 \\ -1 & 1 \end{pmatrix} \rightsquigarrow \alpha = \begin{pmatrix} d & -1 \\ 0 & 1 \end{pmatrix} \qquad \therefore Cl(U_\sigma) \simeq \mathbb{Z}_d$$

Example For $X = \mathbb{P}^n$

$$0 \to \mathbb{Z}^n \to \mathbb{Z}^{n+1} \to \mathbb{Z} \to 0 \qquad \therefore Cl(\mathbb{P}^n) = \mathbb{Z}$$

$$\begin{pmatrix} -1 & \cdots & -1 \\ 1 & \cdots & 0 \\ 0 & \ddots & \\ \vdots & & \\ 0 & \cdots & 1 \end{pmatrix} \qquad (1, 1, \cdots, 1)$$

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From now on, we denote by $\mathrm{Div}_{T_N}(X_\Sigma) := \bigoplus \mathbb{Z} D_\rho$

Let $\mathrm{CDiv}_{T_N}(X_\Sigma) \subseteq \mathrm{CDiv}(X_\Sigma)$

Thm $(0 \to)\ M \xrightarrow{\alpha} \mathrm{CDiv}_{T_N}(X_\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0$ exact.

α is inj $\Leftrightarrow$ $\{u_\rho\}$ spans $N_{\mathbb{R}}$

↳ Similar to Cl's case.

Prop $\mathrm{Pic}(U_\sigma) = 0$

↳ (idea of proof) Pick $D = \sum a_\rho D_\rho$: T_N-inv. Cartier

- We may assume $D \geq 0$ ($\because$ by adding $\mathrm{div}(\chi^{km})$ for $m \in \mathring{\sigma}^\vee$ and $k \gg 0$)
- Pick $p \in O(\sigma)$ ~ minimal orbit

  $\rightsquigarrow p \in D_\rho$ for any $\rho \in \sigma(1)$

- Let $U \subseteq U_\sigma$: open nbd of p s.t. $D|_U = \mathrm{div}(f)|_U$ ≥ 0 for some $f \in \mathbb{C}(U)$. Since "effectiveness", $f \in \mathbb{C}[U]$

  If $U_\sigma \setminus U = V(f_1, \cdots, f_k)$, letting $h = f_1 \cdots f_k$, $(U_\sigma \setminus U = V(h))$

  $\mathbb{C}[U] \simeq \mathbb{C}[\sigma^\vee \cap M]_h \ni f$

  Since $\mathrm{div}(h)|_U = 0$, $\mathrm{div}(f \cdot h^k)|_U = \mathrm{div}(f)|_U$

  $\therefore$ We may assume $f \in \mathbb{C}[\sigma^\vee \cap M] =: R$

$\rightsquigarrow \mathrm{div}(f) \geq 0$

1°

- $\operatorname{div}(f) = \sum_\rho \nu_\rho(f) D_\rho + \sum_{E \neq D_\rho} \nu_E(f) E \geq \underbrace{\sum_\rho \nu_\rho(f) D_\rho}_{= D \ (*)}$

(*): $\operatorname{div}(f)\big|_U = D\big|_U = \sum_\rho a_\rho \cdot (D_\rho \cap U)$ non-empty for $\forall \rho$. (containing p)

$\therefore \nu_\rho(f) = a_\rho$ for $\forall \rho$.

$\therefore \operatorname{div}(f) \geq D$

- Set $\underbrace{\Gamma(U_\sigma, \mathcal{O}_{U_\sigma}(-D))}_{=: I} := \{ f \in \mathbb{C}(U_\sigma) : f = 0 \text{ or } \operatorname{div}(f) - D \geq 0 \}$

$\subseteq R$ ($\because D$ is effective)

"ideal of R"

Also, I is T_N-inv. ($\because D$ is T_N-inv.)

$\therefore I \simeq \bigoplus_{\operatorname{div}(\chi^m) \geq D} \mathbb{C} \cdot \chi^m$ ↪ You should check that

$\mathbb{C}[\sigma^\vee \cap M] \simeq \bigoplus_{m \in \sigma^\vee \cap M} \mathbb{C} \cdot \chi^m$

as T_N-rep.

$\therefore f = \sum a_i \chi^{m_i}$ for m_i with $\operatorname{div}(\chi^{m_i}) \geq D \quad \forall i$.

- $\operatorname{div}(\chi^{m_i})\big|_U \geq D\big|_U = \operatorname{div}(f)\big|_U \quad \therefore \dfrac{\chi^{m_i}}{f}$: morphism on U

$\rightsquigarrow 1 = \sum a_i \cdot \dfrac{\chi^{m_i}}{f}$ implies that $\dfrac{\chi^{m_i}}{f}(p) \neq 0$ for some i.

$\rightsquigarrow$ Let $V \subseteq U$ open s.t. $\operatorname{div}\left(\dfrac{\chi^{m_i}}{f}\right) = 0$ i.e., nowhere vanishing

Then $p \in V \subseteq U$ & $\operatorname{div}(\chi^{m_i})\big|_V = \operatorname{div}(f)\big|_V = D\big|_V$

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- Conclude that $\mathrm{div}(\chi^{m_i}) = D$ $\quad (\because\ \rho \in V \cap D_\rho\ \forall \rho)$

$= \sum_\rho \langle m_i, u_\rho \rangle D_\rho$

Q.E.D

Ex. Previous example $\sigma = \mathrm{cone}(de_1 - e_2, e_2)$ $d > 1$

$Cl(U_\sigma) = \mathbb{Z}_d$

$Pic(U_\sigma) = 0$

Defn. $D \in Div(X)$ is $\mathbb{Q}$-Cartier if $kD \in CDiv(X)$ for some $k \in \mathbb{Z}_{>0}$.

Prop (TFAE)
(a) $\forall$ Weil is Cartier $\quad$ $\mathbb{Q}$-Cartier
obvious (b) $Pic = Cl$ $\quad$ $Pic \le Cl$ finite index
(c) X_Σ smooth $\quad$ X_Σ simplicial.

(C) → (a) smooth affine = $\mathrm{Spec}(R)$, R : UFD.

$\rightsquigarrow D_\rho = \mathbb{V}(f)$ for some $f \in R = \mathbb{C}[U_\sigma]$

$\rightsquigarrow D_\rho = \mathrm{div}(f)$.

(A) → (c) $Pic(U_\sigma) = Cl(U_\sigma) = 0$

$$\rightsquigarrow M \xrightarrow{\text{surj}} \bigoplus_{\rho \in \sigma(1)} \mathbb{Z} D_\rho \to \cancel{Cl(U_\sigma)}^{\,0} \to 0$$

$m \mapsto \sum \langle m, u_\rho \rangle D_\rho$

$\therefore \{u_\rho\}_{\rho \in \sigma(1)}$: part of $\mathbb{Z}$-basis of N

2 Cartier data and support ftns

Thm (TFAE) Let $D = \sum a_\rho D_\rho$: Weil divisor

(a) D : Cartier

(b) For $\forall \sigma \in \Sigma$, $\exists\, m_\sigma \in M$ s.t. $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for $\forall \rho \in \sigma(1)$

(c) For $\forall \sigma \in \Sigma_{max}$, " " "

unique when σ : full-dim'l.

(a) $\leftrightarrow$ (b) or (c) We have seen that $D|_{U_\sigma} = \mathrm{div}(\chi^{-m_\sigma})|_{U_\sigma}$ in the proof of $\mathrm{Pic}(U_\sigma)=0$

for some $m_\sigma \in M$

Defn $\{m_\sigma\}_{\sigma\in\Sigma}$ is called **Cartier data**

Let $P \subseteq M_{\mathbb{R}}$: full dim'l lattice polytope

$= \{ m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F, \ \forall \text{ facet } F \}$

Set $D_P := \sum a_F D_F$

Prop D_P is Cartier with Cartier data $\{ v : \text{vertex of } P \}$

$\updownarrow$

$(\because \langle v, u_F \rangle = -a_F$ for $\forall$ facet $F \ni v)$ $\quad \sigma_v$: max. dim'l cone

* **Question** : If $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$, we may define

$$P_D := \{ m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -a_\rho \}$$

Is it a polytope? (even Σ : complete) No in general.

So, instead of using polytopes, we introduce the following notion

Defn. Given fan Σ in $N_{\mathbb{R}}$.

(a) $\varphi : |\Sigma| \to \mathbb{R}$ support ftn if it is linear on each σ.

(b) φ is integral wrt N if $\varphi(|\Sigma| \cap N) \subseteq \mathbb{Z}$

Denote by $SF(\Sigma)$, $SF(\Sigma, N)$ set of supp. ftns (int wrt N). resp.

Thm Let $D = \sum_\rho a_\rho D_\rho$ Cartier with Cartier data $\{m_\sigma\}$

(a) Define $\varphi_D : |\Sigma| \to \mathbb{R}$

$$u \mapsto \langle m_\sigma, u\rangle \text{ if } u \in \sigma$$

Then $\varphi_D \in SF(\Sigma, N)$

(b) $a_\rho = -\varphi_D(u_\rho)$ so that $D = \sum_{\rho\in\Sigma(1)} -\varphi_D(u_\rho) D_\rho$

(c) $CDiv_{T_N}(X_\Sigma) \simeq SF(\Sigma, N)$

↳ (a), (b) obvious.

$$CDiv_{T_N}(X_\Sigma) = \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}D_\rho \to SF(\Sigma, N) \quad \text{is injective by (b).}$$

$$D \mapsto \varphi_D$$

(Note $\{m_\sigma^D\}$, $\{m_\sigma^E\}$ Cartier data of D, E

then $\{m_\sigma^D + m_\sigma^E\}$ Cartier data of $D+E$

$\{k m_\sigma^D\}$ " of kD)

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Surjectivity : For $\varphi \in SF(\Sigma, N)$, fix $\sigma \in \Sigma$

Then

$\varphi|_\sigma$: linear map on $\sigma \cap N$ to $\mathbb{Z}$.

$\varphi|_\sigma$: $\underset{\mathbb{N}}{\wedge}$ linear

N_σ : span of it

$\therefore \varphi|_\sigma = \langle m_\sigma, \ \rangle$ for some $m_\sigma \in M_\sigma$

$M_\sigma = M / \sigma^\perp \cap M$

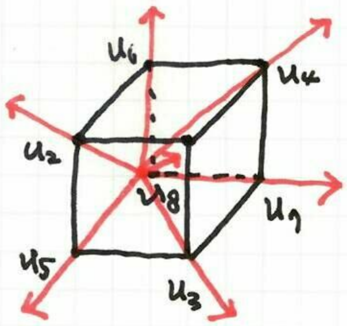
$\rightsquigarrow \{m_\sigma\}$: Cartier data

for $D = \sum_\rho -\varphi(u_\rho) D_\rho$.

Example. (Complete X_Σ but non-projective)

$\operatorname{Pic} X_\Sigma = 0$.

Consider the cube with vertices $(\pm 1, \pm 1, \pm 1)$ in $N_{\mathbb{R}} = \mathbb{R}^3$



Replace $(1,1,1)$ with $u_1 = (1,2,3)$, we get a fan with 6 3-dim'l cones.

Take any $\varphi \in SF(\Sigma, N)$.

- Adding some globally linear ftn integral wrt N, we may assume

$$\varphi(u_1) = \varphi(u_2) = \varphi(u_3) = \varphi(u_5) = 0$$

Some relations : $u_2 + u_8 = u_5 + u_6 \rightsquigarrow \varphi(u_6) = \varphi(u_8)$

$u_6 + u_7 = u_4 + u_8 \rightsquigarrow \varphi(u_4) = \varphi(u_7)$

$u_5 + 2u_7 = u_3 + u_8 \rightsquigarrow \varphi(u_7) = \varphi(u_8)$

But, for 1473 : $4\varphi(u_7) = 5\varphi(u_4)$ $\therefore 0$.

u_1 1 2 3
u_4 −1 1 1
u_3 1 1 −1
u_7 −1 1 −1

$2u_1 + 4u_7 = 3u_3 + 5u_4$

$\therefore \operatorname{Pic}(X_\Sigma) = 0$

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3 Sheaf of a divisor

Defn. A structure sheaf $\mathcal{O}_X$ on a variety X is defined by

$$\mathcal{O}_X(U) := \{ \phi \in \mathbb{C}(X) \mid \phi \text{ is defined on } U \}$$

commutative ring with 1

or $\phi|_U$ is regular

$(\Leftrightarrow \operatorname{div} \phi|_U \geq 0)$

• Let $D = \sum a_p D_p \in \operatorname{Div} X$. Define $\mathcal{O}_X(D)$: sheaf of $\mathcal{O}_X$-module

$$\mathcal{O}_X(D)(U) := \{ f \in \mathbb{C}(X)^* \mid (\operatorname{div}(f) + D)|_U \geq 0 \} \cup \{0\}$$

having at most simple pole along D

Prop. If $D \sim E$, then $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E)$

$(D = E + \operatorname{div}(g))$

$$\begin{aligned} \circ \quad \operatorname{div}(f) + D &= \operatorname{div}(f) + E + \operatorname{div}(g) \\ &= \operatorname{div}(fg) + E \end{aligned}$$

$\therefore f \underset{\simeq}{\mapsto} fg$ isomorphism of section spaces.

Prop. $\mathcal{O}_X(D)$: coherent sheaf of X

$X = \bigcup_\alpha U_\alpha$, U_α : affine $(U_\alpha = \operatorname{Spec} R_\alpha)$

s.t $\mathcal{F}(U_\alpha) \simeq \widetilde{M}_\alpha$ for some $\underbrace{R_\alpha\text{-module}}_{\text{fin.gen.}}$ M_α

Now we come back to toric world. Let $D = \sum a_\rho D_\rho \in \mathrm{Div}\, X_\Sigma$

Prop. $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus\limits_{\mathrm{div}(\chi^m)+D \geq 0} \mathbb{C}\cdot \chi^m$

↳ For $f \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$,

- $\mathrm{div}(f) + D \geq 0$ implies that $\mathrm{div}(f)\big|_{T_N} \geq 0$ $(\therefore f \in \mathbb{C}[M])$

  $\rightsquigarrow \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \subseteq \mathbb{C}[M]$

- $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ is T_N-invariant.
  (T_N-rep.)

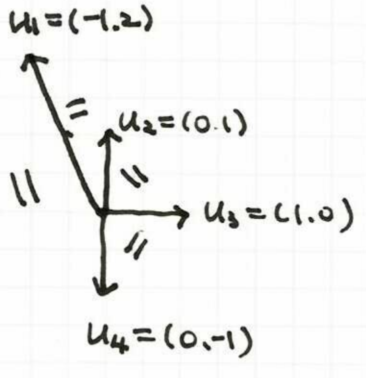
Q.E.D.

Consider the set $\{ m \in M \mid \underbrace{\mathrm{div}(\chi^m)}_{\sum_\rho (\langle m, u_\rho\rangle + a_\rho) D_\rho} + D \geq 0 \}$

epual to $P_D \cap M$ where

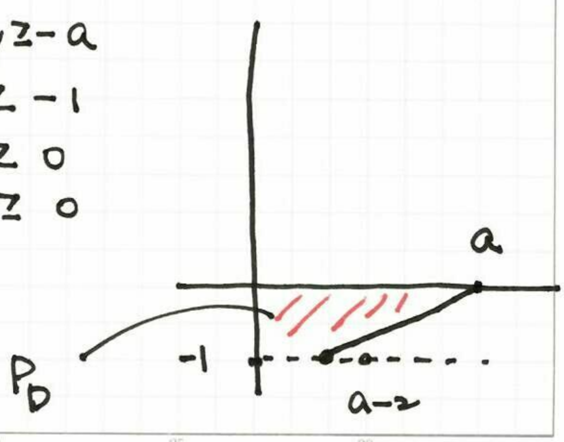
$$P_D := \{ m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle + a_\rho \geq 0 \}$$

Ex



Let $D = aD_1 + D_2$ $(a \in \mathbb{Z})$

$-x+2y \geq -a$
$y \geq -1$
$x \geq 0$
$-y \geq 0$



Hirzebruch surface $\mathcal{H}_2$

- $a > 2 \rightsquigarrow \Sigma_{P_D} = \Sigma$
- $a = 1$ not lattice polytope.
- $a = 2 \rightsquigarrow$

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Hw. $P \subseteq M_{\mathbb{R}}$ full dim'l lattice polytope with

$$P = \{ m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F \}$$

$\rightsquigarrow$ we get X_P and $D_P = \sum a_F D_F$

(a) $P_{D_P} = P$

(b) $\Gamma(X_P, \mathcal{O}_{X_P}(D_P)) = \bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m$

More generally, $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m$

$\boxed{4}$ Line bundles and Cartier divisors.

- From Cartier to line bundle : let $\{(U_i, f_i)\}$: local data.

$D|_{U_i} = \operatorname{div}(f_i)|_{U_i}$

Since $\operatorname{div}(f_i)\big|_{U_i \cap U_j} = \operatorname{div}(f_j)\big|_{U_i \cap U_j} = D\big|_{U_i \cap U_j}$,

$\operatorname{div}(f_i/f_j) = 0$ on $U_i \cap U_j$, i.e., $\underbrace{f_i/f_j}_{=: g_{ij}} : U_i \cap U_j \to \mathbb{C}^*$

$\therefore$ This gives a gluing map

$$\begin{tikzcd} U_i \cap U_j \times \mathbb{C} \arrow[r, "1 \times g_{ij}"] \arrow[d, phantom, "\cap"] & U_i \cap U_j \times \mathbb{C} \arrow[d, phantom, "\cap"] \\ U_j \times \mathbb{C} & U_i \times \mathbb{C} \end{tikzcd}$$

check $g_{ij} \circ g_{jk} = g_{ik}$.

$\rightsquigarrow$ gives a line bundle L.

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- Conversely, any line bundle $\mathcal{L}$ induces a
  "locally free sheaf of rank 1"
  $^{\exists}\{U_\alpha\}$ s.t $\mathcal{F}(U_\alpha) \simeq \underbrace{\mathcal{O}_X(U_\alpha)}$
  space of sections of $U_\alpha \times \mathbb{C} \to U_\alpha$

Let $\mathcal{O}_{\mathcal{L}}$: sheaf of sections of $\mathcal{L}$.

If $\mathcal{L} = \mathcal{L}_D$, then $\mathcal{O}_{\mathcal{L}} \simeq \mathcal{O}_X(D)$.

Some notions & theorems

Suppose X : complete, D: Cartier, $\mathcal{L} := \mathcal{O}_X(D)$

Defn. (a) D : basept-free if $^{\forall} p \in X$, $^{\exists}$ global seltion $s \in \mathcal{O}_X(D)(X)$ s.t. $s(p) \neq 0$.

P_D : lattice polytope $\rightsquigarrow$ $X \xrightarrow[\phi_D]{} \mathbb{P}(H^0(X, \mathcal{L})^*)$ is well-defined.

(b) D is very ample ϕ_D is an embedding

(c) D is ample if kD is very ample for $k \gg 0$

$\Sigma_{P_D} = \Sigma$.

(d) $K_X := \sum_\rho D_\rho$ (every $a_\rho = 1$) is anti-canonical divisor.

(e) X : Fano if K_X is ample.

$\boxed{5}$ Divisors on X_Σ

- $\rho \in \Sigma(1) \rightsquigarrow O(\rho) \rightsquigarrow \underbrace{V(\rho) = \overline{O(\rho)}}_{=: D_\rho}$ ← prime divisor. why?

- For $m \in M$, $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^* \subseteq \mathbb{C}$ is a rational ftn on X_Σ

How to compute $\nu_{D_\rho}(\chi^m)$? $\quad (X \setminus T_N = \bigcup_\rho D_\rho)$

Proposition $\nu_\rho(\chi^m) = \langle m, u_\rho \rangle$ ⇝ show this by example $\mathbb{C}^2$

↳ Enough to consider it in $U_\rho \underset{\text{open}}{\subseteq} X_\Sigma$ (any open set U with $U \cap D_\rho \neq \emptyset$)

u_ρ : primitive. $\underset{\text{extend } \mathbb{Z}\text{-basis}}{\rightsquigarrow}$

$$e_1 = u_\rho, e_2, \cdots, e_n \text{ of } N$$

$$w_1 \cdots w_n \text{ dual basis of } M.$$

$\mathbb{C}[\rho^\vee \cap M] = \mathbb{C}[x_1, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]$ & $U_\rho \simeq \mathbb{C} \times (\mathbb{C}^*)^{n-1}$

two orbits T_N & $O(\rho)$

$U_\rho \cap D_\rho = \{0\} \times (\mathbb{C}^*)^{n-1}$

If $\chi^m = x_1^{a_1} \cdots x_n^{a_n}$, then

$$\nu_\rho(\chi^m) = a_1 = \langle m, u_\rho \rangle$$

Cor $\operatorname{div}(\chi^m) = \sum_\rho \langle m, u_\rho \rangle D_\rho$

Thm $M \xrightarrow{\alpha} \underbrace{\bigoplus \mathbb{Z} D_\rho}_{\text{torus inv divisors}} \to Cl(X_\Sigma) \to 0$ exacts

$\cancel{Cl(T_N)} = 0$

$\alpha(m) := \operatorname{div}(\chi^m)$

Also, α is inj. iff X_Σ has no torus factor

- $\alpha(m) = 0 \Leftrightarrow \langle u_\rho, m \rangle = 0 \ \forall \rho$.
- $\sum a_\rho D_\rho = \operatorname{div}(f) \Leftrightarrow f \in \mathbb{C}[M]^* \Leftrightarrow f = c \cdot \chi^m$